Sensitivity Analysis

Recall the model we have been examining.

\[
\frac{dx}{dt}(t) = f(x(t), p)
\]

where \(p\) is a vector of \(N_p\) parameters (i.e. it is the parameter set) and \(x\) is a vector of \(N_s\) state variables.

We have talked about local sensitivity analysis in which we determine the sensitivity of a cost function to changes in parameters, i.e. Given a function \(c(p)\) of the solution to Eq. 1 for parameter set \(p\) (which is an \(N_p \times 1\) vector), we want to know how it changes as the \(j\)th parameter changes.

\[
\frac{\partial c}{\partial p_j} = \lim_{\Delta p_j \to 0} \frac{c(p^*) - c(p)}{\Delta p_j}
\]

where \(p^* = p + e_j \Delta p_j\) and \(e_j\) is the \(j\)th row of the \(N_p \times N_p\) identity matrix.

Today, we want to examine the sensitivity of the state trajectories over time, i.e.

\[
\frac{\partial x_i}{\partial p_j}(t) = \lim_{\Delta p_j \to 0} \frac{x_i(t, p^*) - x_i(t, p)}{\Delta p_j}
\]

This measure will tell us how the trajectories of the perturbed model will drift away from the trajectories of the unperturbed model.

To compute these new sensitivities, we will use the same approach we used on Tuesday - i.e. we use a finite difference approximation of the partial derivative. We run the simulation with the unperturbed parameter set, then run it with the perturbed set, and compute the difference.

We need to ensure that both simulations begin at the same initial conditions. i.e. \(x(0, p^*) = x(0, p)\). Further, it makes sense to measure deviations from \(x(t, p)\) only if \(x(0, p)\) is already on the limit cycle.
Here is the algorithm:

- Simulate the reference trajectory $x(t, p)$ and grab a value from some time late in the trajectory, after it has found the limit cycle.
- Simulate the reference trajectory from that new starting point.
- For each parameter:
  - Perturb that parameter.
  - Simulate the perturbed trajectory from the same starting point.
  - Compute the sensitivity of all states to this parameter.

On Tuesday, we talked about making our sensitivities relative the parameter values. This makes it reasonable to compare the sensitivity of the system to $p_i$ with the sensitivity to $p_j$. Now we need to add another factor – we need to to be reasonable to compare the sensitivity of state $x_i$ to state $x_j$. so we should actually compute

$$S_{ij}^{rel}(t) = \frac{p_j \frac{\partial x_i}{\partial p_j}(t)}{x_i \frac{\partial p_j}{\partial p_j}(t)}$$