

Assignment 10 Solutions

Recall the statement of the decision version of the max flow problem:

FLOW

Input: A graph G with two marked vertices s and t , where every edge e has a capacity c_e , and a parameter k .

Solution: TRUE if there is a flow of value at least k , and FALSE otherwise.

Consider the following flow problem with demands:

FLOW WITH DEMANDS (FWD)

Input: A graph G , a capacity c_e for each edge e , and a demand d_v for every vertex v .

Solution: TRUE if there is a valid flow such that $f^{\text{out}}(v) - f^{\text{in}}(v) = d_v$ for every vertex v , and FALSE otherwise.

Our goal is to show $\text{FWD} \leq \text{FLOW}$.

1. What does that previous sentence mean?

Solution: It means we can take an input G, c, d for the FWD problem and compute an *equivalent* input G', c', s, t for the FLOW problem. That is, there is a valid flow on G satisfying the constraints given by c and demands given by d if and only if there is a flow on G' satisfying the capacities given by c' where s is a source node and t is a sink node in G .

2. Inputs of which problem are we taking, and input of which problem are we outputting?

Solution: We are taking an input for FWD, and computing an equivalent input for FLOW.

3. Design an algorithm that computes the reduction (hint: create a source and sink. For every vertex with negative demand, create an edge from the source to it, for every vertex with positive demand create an edge from it to the sink).

Solution: First, let S be the sum of all demands in the graph. Note that for a valid flow to be possible in G , it better be the case that $S = 0$ (this part left as something for the reader to think about).

If $S \neq 0$, we create G' by just having two vertices, namely s and t , without any edges and set $k = 1$.

If $S = 0$, then to create G' , we keep all the same vertices in G but add two new vertices s and t . We add edges from s to every vertex v with negative demand and set $c'_{(s,v)} = |d_v|$. We add edges from every vertex v with positive demand to t with capacity $c'_{(v,t)} = d_v$. For all other edges e in the original graph G , we set $c'_e = c_e$. Lastly, we set k to be the sum of the capacities of edges leaving s . Note that since $S = 0$, this is the same as the sum of the capacities into t .

4. Convince me this reduction works.

Solution: First, if there is a valid flow in G , consider the same flow on G' where we also completely saturate all edges out of s and all edges into t with flow. Then for any v such that $d_v < 0$, we have that $f_G^{\text{in}}(v) - f_G^{\text{out}}(v) = d_v$. Since we have d_v flow coming

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from s to v in G' we have that $f_{G'}^{\text{in}}(v) - f_{G'}^{\text{out}}(v) = f_G^{\text{in}}(v) - f_G^{\text{out}}(v) + |d_v| = 0$. The same applies for vertices v with $d_v > 0$. Lastly, since we kept all capacities the same in G' , we aren't breaking any capacity constraints. Thus we have a valid flow that saturates all edges out of s , giving us a flow with value k .

On the other hand, if there is valid flow f' in G' of value k , then we must be in the case where $S = 0$ because otherwise we would have had the silly bad instance we described with just s and t and no edges. Note that to have a valid flow we must be saturating all edges out of s and all edges into t . Consider the same flow f as restricted to the graph G : for every vertex v with negative demand we've just taken out the supply coming from s , which must have been all $|d_v|$ since the edge must have been fully saturated as described above. Hence $f_G^{\text{out}}(v) - f_G^{\text{in}}(v) = d_v$. The same applies for vertices v with $d_v > 0$. Vertices v with $d_v = 0$ have no connection to s or t , so $f_G^{\text{out}}(v) - f_G^{\text{in}}(v) = 0$. Again, since all capacities of edges in G were kept the same, we aren't breaking any edge capacity constraints with our flow f , and hence we have a valid flow that satisfies the demands of all the vertices in the graph.

Hence these inputs are equivalent, and thus our reduction is correct.