

CS381 Assignment 3

Due October 19 2023 11:59 PM AoE

1. Recall the *Local Connection Game* we studied in class: in this game, the n players are modeled as vertices who have the option of buying edges to other players/vertices at the cost of α per edge. The perceived resultant cost c_i of player i is the summed costs of their edges they paid for and their perceived distances to every other vertex (if a vertex cannot be reached, their perceived distance is infinite). Specifically,

$$c_i = \alpha E_i + \sum_{j \neq i} \text{dist}(v_i, v_j),$$

where E_i is the number of edges purchased by player i and v_x denotes the vertex corresponding to player x . We defined the social cost c_G of a graph G to be

$$\sum_i c_i = \alpha E + \sum_{\text{all pairs } (i,j)} \text{dist}(v_i, v_j),$$

where E is the total number of edges purchased by all players. For n players, let G_n^* denote the graph that minimizes the social cost function (in class, we showed that if $\alpha \leq 2$ then G_n^* is a complete graph, and if $\alpha \geq 2$ then G_n^* is a star graph).

In class, we established the following two lemmas:

Lemma 1. For a graph G produced by n players in a state of Nash Equilibrium, we have $\frac{c_G}{c_{G_n^*}} \leq d_G$ where d_G is the diameter of G .

Lemma 2. The diameter d_G of a graph G produced by players in a state of Nash Equilibrium satisfies $d_G = O(\sqrt{\alpha})$.

Together, these two lemmas immediately gave us the theorem:

Theorem 1. The Price of **Anarchy** of the Local Connection Game is $O(\sqrt{\alpha})$.

Our goal in this question will be to improve this a bit.

- (a) First, notice that if α grows unboundedly, the bound above for the Price of **Anarchy** is also unbounded. Let's fix that; specifically, let's show that for $\alpha > n^2$, we have that the Price of **Anarchy** is $O(1)$.
 - i. If $\alpha > n^2$, show that any **connected** graph with a cycle must not be a Nash Equilibrium.
 - ii. **Extra Credit: (Shout out to Clio):** Even if $\alpha > n^2$, not every graph without a cycle is a Nash Equilibrium.

- iii. If $\alpha > n^2$, show that the price of **Anarchy** is $O(1)$. Hints etc: You can assume $n \geq 2$, which means $\alpha > 4$, so the optimal solution is a star; the game is pretty dull with $n = 1$.

Do either b or c below or both for some amount of extra credit and my adoration (I think b is actually much easier, and if your initials are ZG and are still interested in AIT you must do b):

- (b) Let's tighten up Lemma 2 above: specifically, let's show that $d_G = O(1 + \frac{\alpha}{\sqrt{n}})$. Throughout this problem, we'll be assuming G is a graph produced by n players in a state of Nash Equilibrium. We already know that for small diameter Nash Equilibria, the price of **Anarchy** is effectively constant (by Lemma 1 above), so let's try and deal with bigger diameters.

Let $d' = \lfloor \frac{d-1}{4} \rfloor$, ie the result returned by Java's integer division when I divide $d - 1$ by 4, that is the value received by calculating $\frac{d-1}{4}$ and deleting the non-integer part. Let u and v be vertices such that $\text{dist}(u, v) = d$. Let B be the collection of vertices at most d' distance from u . We want to investigate the choice of players u and v to not buy more edges to save themselves cost in perceived distance. Let w be any vertex in B .

- i. Under the assumption that d is reasonably big, confirm $\alpha > d > d'$. (just use Lemma 2 above)
 - ii. Confirm that $\text{dist}(w, v) \geq d - d'$.
 - iii. If v were to buy the edge to player u , confirm that $\text{dist}(w, v) \leq d' + 1$.
 - iv. Confirm that $|B|^{\frac{d-1}{2}} \leq \alpha$ where $|B|$ is the size of the collection B .
 - v. Suppose $\text{dist}(u, w) = d'$ and let A_w be the collection of vertices x for which the shortest path from u to x goes through w and immediately exits B (to a vertex that is distance $d' + 1$ from u).
Confirm that if u bought the edge to w they would reduce their costs due to perceived distances by at least $|A_w|(d' - 1)$, where $|A_w|$ is the number of items in A_w .
 - vi. Confirm that $|A_w|(d' - 1) \leq \alpha$.
 - vii. Confirm that there must be a vertex w in B where $|A_w| \geq \frac{n - |B|}{|B|}$. Hint: Pigeon Hole Principle
 - viii. Combine your above inequalities to get $\alpha^2 \geq n(d' - 1)^2$ (probably the most annoying step, pretty much just comparing big and small things above).
 - ix. Show that the Price of **Anarchy** is $O(1 + \frac{\alpha}{\sqrt{n}})$.
- (c) This question investigates the rate of convergence for the local congestion game on random graphs where everyone uses best responses to determine whether they buy any new edges or give up any currently owned edges.
- i. Create a program that creates random graphs of n vertices according to the following process: given some probability p , for each (unordered) pair of vertices u and v , create an edge between them with probability p . If an edge is created, set u to be the 'buyer' with 50% probability and otherwise set v to be the owner.
 - ii. Loop through the players cyclically and have each player play their best response to the current state until the graph is in a state of Nash Equilibrium. Record the number of iterations (1 iteration = 1 player's turn) it took to reach equilibrium.
 - iii. Letting $n = 40$, as a function of p and α analyze how many iterations on average it takes to reach equilibrium. I would expect a significantly large number of trials to be run to create data to back up whatever claims you make in your analysis.

- iv. **Extra Credit:** Show that the Price of Anarchy in this game is $O(1)$ regardless of α .
Hint: Have an insight that every person who has ever looked at this problem so far has not had.
2. Recall the Shapley Network Design game from lecture: in this problem, we are given a directed graph G where each edge e has an associated cost c_e . Each player i has an associated pair of vertices s_i and t_i and must choose a path in the graph from s_i to t_i . The cost c_i player i receives is their share of the edges they use in their path P_i :

$$c_i = \sum_{e \in P_i} \frac{c_e}{k_e}$$

where k_e is the number of players whose path includes traversing e . The social cost of a solution C_S (S can be thought of as a specification of a collection of edges that connects all players s/t pairs) is

$$C_S = \sum_{e \in S} c_e = \sum_{\text{players } i} c_i$$

- (a) In class we showed that the Price of Anarchy of this game is at least n , where n is the number of players. In fact, it can never exceed n .
- i. Let P_i^* be the minimal cost path from s_i to t_i if this were a 1 player game for just player i (equivalently, P_i^* is the minimal cost path from s_i to t_i if costs were not shared by players). Confirm that in a Nash Equilibrium state,

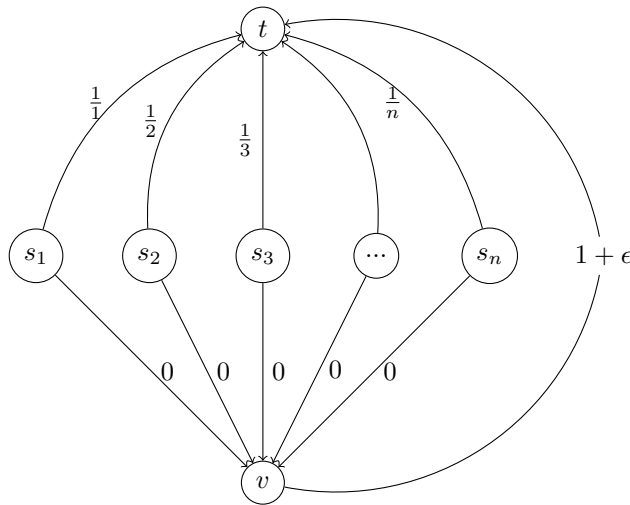
$$c_i \leq \sum_{e \in P_i^*} c_e.$$

- ii. Confirm that

$$\sum_{e \in P_i^*} c_e \leq c_{\text{OPT}}$$

where OPT is the minimal cost solution.

- iii. Confirm that the Price of Anarchy of this game is at most n . Hint: take the above and sum over all the players.
- (b) In class we showed that the price of stability is at most $H_n = \sum_{i=1}^n \frac{1}{i}$. We'll now show that the price of stability is at least H_n . Let $\epsilon > 0$ (imagine it's infinitesimally small though) and consider the following graph:



In the above graph, $t = t_1 = t_2 = \dots = t_n$, so every player has the same target.

- i. Confirm that there is exactly one (pure) Nash Equilibrium in this graph.
- ii. Confirm that the Price of Stability of this game is H_n .
- iii. Confirm that if this graph's edges were replaced with undirected edges, the price of stability of this specific graph would be 1.
- iv. **Extra Credit:** It is known that the Price of Stability for this game when the graphs are undirected is $O(\frac{\log n}{\log \log n})$, but a matching lower bound has not been found. Establish that the Price of Stability is $O(1)$ or $\omega(1)$ (ω is like Ω , except that it is *strictly greater*. So $\omega(1)$ means that the Price of Stability would be strictly greater than constant: that is, for any constant c given, you can find an instance where the Price of Stability on that instance is greater than c). Hint: Have an insight that every person who has ever looked at this problem so far has not had.