

CS381 Assignment 7

Due December 11 2023 12:00 PM AoE

1. For each of the following cake-cutting algorithms, convince me (a student in 231) that the algorithm is not incentive compatible.
 - (a) For 2 players: the typical “I cut you choose” algorithm.
 - (b) For n players: the proportional moving knife algorithm (a referee moves a knife across the cake - when any player shouts stop, a slice is made where the knife currently is and that player receives the piece).
2. Convince me that the following algorithm (which is basically the natural extension of “I cut you choose”) does not guarantee proportionality for 3 players A B and C:

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Let A make a cut
Let B make a cut
Let C choose their slice
Let B choose their slice
Let A choose their slice
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Do one of 3 or 4

3. In our last class we spent some time trying to show that there was no “good” envy-free mechanism for fair division of a cake. Here, “good” meant that
 - (a) The mechanism was deterministic
 - (b) The cake was divided into n connected pieces (and hence the mechanism made precisely $n - 1$ cuts)
 - (c) The mechanism was finite: here, we only allowed the mechanism to ask players to mark specific %s of the cake (say, $\frac{1}{2}$ or $\frac{2}{3}$ or $\frac{7}{n}$, etc.), and then use that info to decide where to make the cuts.

To show this, we focused on the case where $n = 3$ and we defined the idea of a “Rigid Measurement System (RMS)”. Recall that an RMS for 3 players consists of the parameters x, y, s, t and valuations v where:

- i) $0 < x < y < 1$
- ii) $0 < s < \frac{1}{3} < t < \frac{1}{2}$
- iii) For any piece S of length ℓ , we have $\frac{\ell}{\sqrt{2}} < v_i(S) < \sqrt{2}\ell$

iv) For the following specific pieces, the player valuations v are as follows:

v	$(0, x)$	(x, y)	$(y, 1)$
v_1	s	t	t
v_2	t	s	t
v_3	t	t	s

The power of these RMSs is, as we saw in class, is that they force the cuts to be made precisely at x and y : any other cuts will result in some player being envious of another.

In class we wrote about a wonderful lemma:

Lemma 1. Given a number $\eta > 0$ and an RMS μ with parameters x, y, s, t and valuations v , we can come up with a new RMS μ' with parameters x', y', s', t' and valuations v' such that:

- (a) $x' \neq x$ and $y' \neq y$, but $|x - x'| < \eta$ and $|y - y'| < \eta$
- (b) for all intervals (a, b) where both a and b are at least η far away from x and y we get $v_i(a, b) = v'_i(a, b)$.

Prove this lemma, and explain how we can use it show that no finite mechanism exists for which every player has a strategy that guarantees an envy-free allocation.

4. In class, we talked a little about what I called a divorce problem: we have an even number n of items, where each side has a ranking over the items. We gave an algorithm that decides whether we can guarantee a fair allocation to both players without knowing the literal valuations of the items. That is, it determined whether there was enough information just from the rankings to output a fair allocation if there was one, and something like “Not Enough Info” otherwise.

Given two rankings (ie permutations of $1, \dots, n$) drawn uniformly at random, determine experimentally what the probability is that these two rankings are sufficient to determine a fair allocation. I’m expecting a significant number of experimental trials to determine your answer. From my code (took me about 36 lines), the answer seems to be $1 - \frac{2}{n}$ for large enough n . Which is kinda neat, it means that it’s pretty “unlikely” that there isn’t enough info (although a more reasonable test probably wouldn’t be explicitly uniformly random, and would just add ‘noise’ instead of completely shuffling).

5. We kinda flubbed the proof that the following protocol can guarantee envy-freeness for 3 players:

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Move a knife continuously across the cake until a player P shouts stop.
Mark (don't cut) where the knife currently is.
Have P mark an additional line, designating pieces A B and C (from left to right).
If one of the two remaining players prefers B and the other C:
    Give those two players their preferred pieces from B and C.
    Give P piece A.
Else if they both prefer B:
    Give P a knife.
Continue to move the original knife continuously from A while P
    -- moves the knife inward from the second mark into B
    -- until a player Q != P shouts stop.
Update A, B, and C with the location of the two knives and make the cuts.
Give Q their choice of pieces A or C.
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        Give the remaining player the piece B.
        Give P the remaining piece.
Else (if the both prefer C)
    Give P a knife.
    Continue to move the original knife continuously from A while P
        -- moves the knife outward from the second mark into C
        -- until a player Q != P shouts stop.
    Update A, B, and C with the location of the two knives and make the cuts.
    Give Q their choice of pieces A or B.
    Give the remaining player piece C.
    Give P the remaining piece.

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Let's be a bit more exact and write it cleanly: convince me that *regardless of other players' strategies*, the strategy of playing honestly guarantees the player an envy-free piece.