

1 Definitions

Pure Nash Equilibrium: Every player's strategy is deterministic (no randomness)

Mixed Nash Equilibrium: A Nash Equilibrium where at least one of the players uses randomization in their strategy.

Theorem 2.1. (Nash, 50s) Every game with a finite number of players and strategies for each players has a mixed Nash Equilibrium A mixed strategy Nash equilibrium involves at least one player playing a randomized strategy and no player being able to increase his or her expected payoff by playing an alternate strategy. https://saylordotorg.github.io/text_introduction-to-economic-analysis/s17-03-mixed-strategies.html

2 Finding Mixed Nash Equilibria

- The expected value of a random variable that takes on a collection of possible values is $\sum \text{Probability}(X = x) * x$

- Example: Expected value of 1d6 by counting all possible rolls: $EV = 1 * \frac{1}{6} + 2 * \frac{1}{6} + 3 * \frac{1}{6} + 4 * \frac{1}{6} + 5 * \frac{1}{6} + 6 * \frac{1}{6} = 3.5$
- Example Expected value of 2d6 by counting all possible pairs and their sums: $EV = 2 * \frac{1}{36} + 3 * \frac{2}{36} + \dots + 7 * \frac{6}{36} + 8 * \frac{5}{36} + \dots + 12 * \frac{1}{36} = 7$

- Applying idea to Evens/Odds

	P_2 even	P_2 odd
P_1 even	-1	1
P_1 odd	1	-1

- What happens if P_2 (columns) says even 75% of the time and odd 25% of the time? $\rightarrow P_1$ (rows) best response is to only pick even numbers. Any attempt at randomness only decreases P_1 's chances at winning.
- Claim: For any game, the best response to a player using a mixed strategy is still a pure strategy (up to ties in expected value)
- In Evens/Odds, is both players doing a 50/50 split between even and odd a mixed Nash Equilibrium?
 - From P_1 's perspective:

$$EV(\text{saying even}) = \frac{1}{2} * 1 + \frac{1}{2} * -1 = 0$$

$$EV(\text{saying odd}) = \frac{1}{2} * 1 + \frac{1}{2} * -1 = 0$$

Generally:

$$EV(\text{saying even with probability } q) = \frac{q}{2} * 1 * \frac{1}{2} * -1 + \frac{1-q}{2} * 1 + \frac{1-q}{2} * -1 = 0$$

Thus, P_1 's chances at winning is locked in no matter what they do.

- Is P_1 doing 50-50 and P_2 doing 75-25 a mixed Nash Equilibrium? → No because P_1 can better themselves by changing to an 100-0 strategy.

- Applying to Game of Chicken

	P_2 swerve	P_2 straight
P_1 swerve	0	1
P_1 straight	-1	-100

- Assume that P_2 randomly picks an option with 50/50 chance:

$$EV(P_1 \text{ only swerves}) = \frac{1}{2} * 0 + \frac{1}{2} * -1 = \frac{-1}{2}$$

$$EV(P_1 \text{ only goes straight}) = \frac{1}{2} * 1 + \frac{1}{2} * -100 = -49.5$$

$$EV(P_1 \text{ swerves with probability } q) = \frac{q}{2} * 0 + \frac{q}{2} * -1 + \frac{1-q}{2} * 1 + \frac{1-q}{2} * -100 = -100 \frac{1-q}{2}$$

We can maximize the above equation with $q = 1$, which means P_1 always serves.

- Assume that P_2 swerves with p probability and goes straight with probability $p - 1$:

$$EV(P_1 \text{ serves}) = p * 0 + (1 - p) * -1$$

$$EV(P_1 \text{ goes straight}) = p * 1 + (1 - p) * -100$$

If we combine these two equations and solve for p , we find $p = \frac{99}{100}$. Therefore, there is a mixed Nash Equilibrium if both players swerve with a probability of $\frac{99}{100}$

- Some New Game defined by:

	q	$q - 1$
p	3	2
$p - 1$	2	5
	3	1
	0	6

- We can immediately find 1 pure Nash Equilibrium in the top left but brute force checking every square.
- Lets investigate P_1 options:

$$EV(P_1 \text{ goes } r_1) = q * 3 + (1 - q) * 3 = 3$$

$$EV(P_1 \text{ goes } r_2) = q * 2 + (1 - q) * 5 = 5 - 3q$$

$$EV(P_1 \text{ goes } r_3) = q * 0 + (1 - q) * 6 = 6 - 6q$$

Notice that with $q = \frac{2}{3}$ the expected value for r_1 and r_2 is 3.

- Repeat process for P_2 , focusing on rows 1 and 2:

$$\text{EV}(P_2 \text{ goes } c_1) = p * 3 + (1 - p) * 2 = p + 2$$

$$\text{EV}(P_2 \text{ goes } c_2) = p * 2 + (1 - p) * 6 = 6 - 4p$$

Notice that the expected values are the same with $p = \frac{4}{5}$. We have found our first mixed Nash Equilibrium with $q = \frac{2}{3}$ and $p = \frac{4}{5}$!

- This algorithm is very slow since it just brute forces every possibility.
- For example of a failed mixed equilibrium, let us try to use r_1 and r_3 :

$$\text{EV}(P_2 \text{ goes } c_1) = p * 3 + (1 - p) * 3 = 3$$

$$\text{EV}(P_2 \text{ goes } c_2) = p * 2 + (1 - p) * 1 = p + 1$$

In order for the expected values to be the same, $p = 2$. Since p is a probability, it must be less than 1, so there is no mixed Nash Equilibrium using rows 1 and 3.

- Now let us try the with rows 2 and 3. Going back to P_1 's options above, we can see that with $q = \frac{1}{3}$ rows 2 and 3 will have the same expected value of 4. Now for P_2 :

$$\text{EV}(P_2 \text{ goes } c_1) = p * 2 + (1 - p) * 5 = 5 - 3p$$

$$\text{EV}(P_2 \text{ goes } c_2) = p * 3 + (1 - p) * 1 = 2p + 1$$

With $p = \frac{4}{5}$, the expected value of both choices is $\frac{13}{5}$. Therefore we have found another Nash Equilibrium using rows 2 and 3 with $p = \frac{4}{5}$ and $q = \frac{1}{2}$

- Finally, let us try to use all three rows. We can try as hard as we want, but there is no q that satisfies both $5 - 3q = 3$ and $6 - 6q = 3$. Therefore, we cannot construct a strategy for P_2 that makes all of P_1 choices the same across all three rows. In other words, we cannot construct a Nash Equilibrium using all 3 rows.