1 Definitions

Pure Nash Equilibrium: Every player’s strategy is deterministic (no randomness)

Mixed Nash Equilibrium: A Nash Equilibrium where at least one of the players uses randomization in their strategy.

Theorem 2.1. (Nash, 50s) Every game with a finite number of players and strategies for each players has a mixed Nash Equilibrium A mixed strategy Nash equilibrium involves at least one player playing a randomized strategy and no player being able to increase his or her expected payoff by playing an alternate strategy. 

2 Finding Mixed Nash Equilibria

- The expected value of a random variable that takes on a collection of possible values is 
  \[ \sum \text{Probability} (X = x) \times x \]
  
  - Example: Expected value of 1d6 by counting all possible rolls: 
    \[ \text{EV} = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = 3.5 \]
  
  - Example Expected value of 2d6 by counting all possible pairs and their sums: 
    \[ \text{EV} = 2 \times \frac{1}{36} + 3 \times \frac{3}{36} + \ldots + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + \ldots + 12 \times \frac{1}{36} = 7 \]

- Applying idea to Evens/Odds

\[
\begin{array}{c|c|c}
\text{P}_1 \text{ even} & \text{P}_2 \text{ even} & \text{P}_2 \text{ odd} \\
\hline
1 & -1 & 1 \\
\hline
-1 & 1 & -1 \\
\end{array}
\]

- What happens if \( \text{P}_2 \) (columns) says even 75% of the time and odd 25% of the time? \( \rightarrow \) \( \text{P}_1 \) (rows) best response is to only pick even numbers. Any attempt at randomness only decreases \( \text{P}_1 \)'s chances at winning.

- Claim: For any game, the best response to a player using a mixed strategy is still a pure strategy (up to ties in expected value)

- In Evens/Odds, is both players doing a 50/50 split between even and odd a mixed Nash Equilibrium?
  
  * From \( \text{P}_1 \)'s perspective:
    
    \[ \text{EV(saying even)} = \frac{1}{2} \times 1 + \frac{1}{2} \times -1 = 0 \]

    \[ \text{EV(saying odd)} = \frac{1}{2} \times 1 + \frac{1}{2} \times -1 = 0 \]
Generally:

\[ \text{EV(saying even with probability } q) = \frac{q}{2} \times \frac{1}{2} \times -1 + \frac{1-q}{2} \times 1 + \frac{1-q}{2} \times -1 = 0 \]

Thus, \( P_1 \)'s chances at winning is locked in no matter what they do.

- Is \( P_1 \) doing 50-50 and \( P_2 \) doing 75-25 a mixed Nash Equilibrium? \( \rightarrow \) No because \( P_1 \) can better themselves by changing to an 100-0 strategy.

- Applying to Game of Chicken

<table>
<thead>
<tr>
<th>( P_1 ) swerve</th>
<th>( P_2 ) swerve</th>
<th>( P_2 ) straight</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( P_1 ) straight</td>
<td>-1</td>
<td>-100</td>
</tr>
</tbody>
</table>

- Assume that \( P_2 \) randomly picks an option with 50/50 chance:

\[ \text{EV}(P_1 \text{ only swerves}) = \frac{1}{2} \times 0 + \frac{1}{2} \times -1 = -\frac{1}{2} \]
\[ \text{EV}(P_1 \text{ only goes straight}) = \frac{1}{2} \times 1 + \frac{1}{2} \times -100 = -49.5 \]
\[ \text{EV}(P_1 \text{ swerves with probability } q) = \frac{q}{2} \times 0 + \frac{q}{2} \times -1 + \frac{1-q}{2} \times 1 + \frac{1-q}{2} \times -100 = -100 \frac{1-q}{2} \]

We can maximize the above equation with \( q = 1 \), which means \( P_1 \) always serves.

- Assume that \( P_2 \) swerves with \( p \) probability and goes straight with probability \( 1-p \):

\[ \text{EV}(P_1 \text{ serves}) = p \times 0 + (1-p) \times -1 \]
\[ \text{EV}(P_1 \text{ goes straight}) = p \times 1 + (1-p) \times -100 \]

If we combine these two equations and solve for \( p \), we find \( p = \frac{99}{100} \). Therefore, there is a mixed Nash Equilibrium if both players swerve with a probability of \( \frac{99}{100} \).

- Some New Game defined by:

\[
\begin{array}{c|cc}
 p & q & q-1 \\
\hline
 3 & 3 & 2 \\
 2 & 5 & \\
\hline
 p-1 & 3 & 1 \\
\end{array}
\]

- We can immediately find 1 pure Nash Equilibrium in the top left but brute force checking every square.

- Lets investigate \( P_1 \) options:

\[ \text{EV}(P_1 \text{ goes } r_1) = q \times 3 + (1-q) \times 3 = 3 \]
\[ \text{EV}(P_1 \text{ goes } r_2) = q \times 2 + (1-q) \times 5 = 5 - 3q \]
\[ \text{EV}(P_1 \text{ goes } r_3) = q \times 0 + (1-q) \times 6 = 6 - 6q \]

Notice that with \( q = \frac{2}{3} \) the expected value for \( r_1 \) and \( r_2 \) is 3.
Repeat process for $P_2$, focusing on rows 1 and 2:

\[
\begin{align*}
\text{EV}(P_2 \text{ goes } c_1) &= p \times 3 + (1 - p) \times 2 = p + 2 \\
\text{EV}(P_2 \text{ goes } c_2) &= p \times 2 + (1 - p) \times 6 = 6 - 4p
\end{align*}
\]

Notice that the expected values are the same with $p = \frac{4}{5}$. We have found our first mixed Nash Equilibrium with $q = \frac{2}{3}$ and $p = \frac{4}{5}$.

This algorithm is very slow since it just brute forces every possibility.

For example of a failed mixed equilibrium, let us try to use $r_1$ and $r_3$:

\[
\begin{align*}
\text{EV}(P_2 \text{ goes } c_1) &= p \times 3 + (1 - p) \times 3 = 3 \\
\text{EV}(P_2 \text{ goes } c_2) &= p \times 2 + (1 - p) = p + 1
\end{align*}
\]

In order for the expected values to be the same, $p = 2$. Since $p$ is a probability, it must be less than 1, so there is no mixed Nash Equilibrium using rows 1 and 3.

Now let us try the with rows 2 and 3. Going back to $P_1$’s options above, we can see that with $q = \frac{1}{3}$ rows 2 and 3 will have the same expected value of 4. Now for $P_2$:

\[
\begin{align*}
\text{EV}(P_2 \text{ goes } c_1) &= p \times 2 + (1 - p) \times 5 = 5 - 3p \\
\text{EV}(P_2 \text{ goes } c_2) &= p \times 3 + (1 - p) = 2p + 1
\end{align*}
\]

With $p = \frac{4}{5}$, the expected value of both choices is $\frac{13}{5}$. Therefore we have found another Nash Equilibrium using rows 2 and 3 with $p = \frac{4}{5}$ and $q = \frac{1}{3}$.

Finally, let us try to use all three rows. We can try as hard as we want, but there is no $q$ that satisfies both $5 - 3q = 3$ and $6 - 6q = 3$. Therefore, we cannot construct a strategy for $P_2$ that makes all of $P_1$ choices the same across all three rows. In other words, we cannot construct a Nash Equilibrium using all 3 rows.