

1 Price of Anarchy and Stability (from last lecture)

To measure the efficiency of Nash Equilibrium, we introduce two definitions:

$$\text{Price of Anarchy: } \max_{\text{equilibriums } E} \left[\frac{\text{Cost of } E}{\text{Best Objective Solution}} \right]$$

Price of Anarchy is the bound for the worst-case Nash Equilibrium against some cost we have to minimize. How bad the game can get according to all possible best solutions.

$$\text{Price of Stability: } \min_{\text{equilibriums } E} \left[\frac{\text{Cost of } E}{\text{Best Objective Solution}} \right]$$

Aim for the best Equilibrium.

The Price of Anarchy is bigger than the Price of Stability.

If there is only **one** equilibrium, the Price of Anarchy equals to the Price of Stability.

Consider the following matrix of cost, where the total cost is the sum of the costs for each player.

	10	100
10	100	1
100	1	100

Nash Equilibriums: bottom right corner (0,0) and top left corner (0,0).

Top left: overall cost = 20.

Bottom right: overall cost = 2.

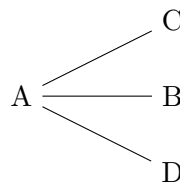
Best objective solution = 2.

Price of Stability = $\frac{2}{2} = 1$

Price of Anarchy = $\frac{20}{2} = 10$

The functions are used in analyzing the games: compared to the case when all the players play altruistically, how bad the outcome can get in a specific game. Through the Price of Stability and the Price of Anarchy, we identify best- and worst-case scenarios for the games, accordingly.

2 The Local Connection Game



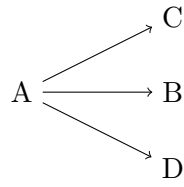
Goal: Connect all players via same path.

Players buy edges, which are then bidirectional and used by everyone.

Cost α to buy an edge.

Player's cost: Sum of money spent on edges + perceived distance in a number of hops to each other player.

Example 1: $n = 4$, $\alpha = 3$



Player A buys 3 edges, while other players use this network.

Example 1: $n = 8$, $\alpha = 3$

Player A buys 3 edges, while other players use this network. No one wants to unilaterally deviate.

Example 2: $n = 8$, $\alpha = 3$

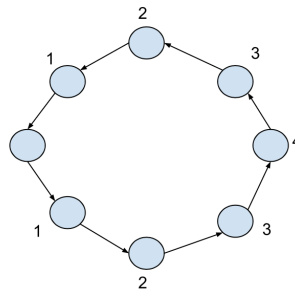


Figure 7.1: A network for 8 players

Everyone buys an edge to the next player. Purchasing an additional blue edge (Figure 7.2) results in saving of \$ 4.

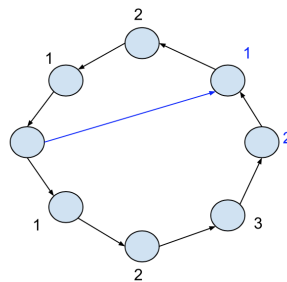


Figure 7.2: A network for 8 players

Objective Cost is the sum of everyone's perceived cost (distances).

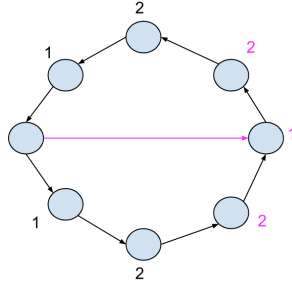


Figure 7.3: A network for 8 players

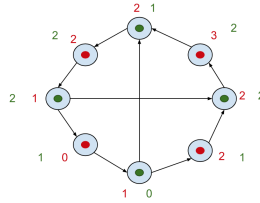


Figure 7.4: A network for 8 players

Cost of Nash Equilibrium: $\$30 + 4 \times 11 + 4 \times 13 = 30 + 44 + 62 = 126$.

Alternative Solution:

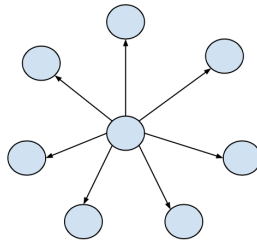


Figure 7.5: A star graph. A network for 8 players

The total cost of such a network: $3 \times 7 + 7 + 7 \times 13 = 119$

This is a Nash Equilibrium. But is it an optimal solution?

Note: when α is big, we want to minimize edges.

Imagine a connected graph, pick a vertex, and do DFS. Then, $n - 1$ edges are necessary to visit every vertex.

For $\alpha \geq 2$, a star graph (Figure 7.5) is an optimal solution.

For $\alpha \leq 2$, a complete graph is an optimal solution (OPT).

Can we come up with a lower bound on the cost of OPT?

The cost of all edges must be at least $\alpha(n - 1)$, just to connect everyone.

Let's say a solution purchases m edges. Then, $\text{Cost} \geq m\alpha + 2m + (n(n - 1) - 2m) \times 2 = m(\alpha - 2) + n(n - 1) \times 2$.

For $\alpha \geq 1$, a star graph is a Nash Equilibrium.
For $\alpha \leq 1$, a complete graph is a Nash Equilibrium.

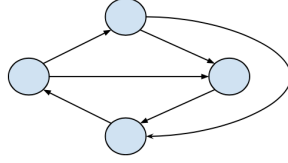


Figure 7.6: A network for 4 players

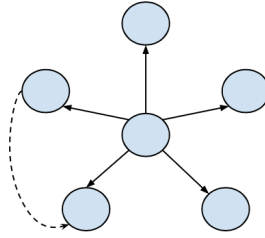


Figure 7.7: A network for 6 players

PoS = 1 for $\alpha \leq 1$ and $\alpha \geq 2$.

What about the middle?

$$\text{PoS} \leq \frac{\text{cost of the star graph}}{\text{cost of the complete graph}} = \frac{(n-1)\alpha + (n-1) + (n-1)(1 + (n-2) \times 2)}{\frac{n(n-1)}{2} \alpha + n(n-1)} = \frac{(n-1)(\alpha + 1 + 1 + (n-2) \times 2)}{(n-1) \times n(\frac{\alpha}{2} + 1)} = \frac{\alpha + 2n - 2}{n(\frac{\alpha}{2} + 1)} \leq \frac{4}{3}$$

(by picking α of 1.

$$\text{PoA} = O(\sqrt{\alpha})$$

Turns out, we can do better! If $\alpha = O(\sqrt{n})$, then $\text{PoA} = O(1)$. Technically, $\text{PoA} = O(1 + \frac{\alpha}{\sqrt{n}})$

If $\alpha = \omega(n \log n)$, then $\text{PoA} = O(1)$.

If a given specification strategies (i.e. a graph G) is a Nash Equilibrium, then the cost of this solution is at most the diameter (*maximal length shortest path between two vertices*) of the graph times OPT.