

1 Local Connection Game:

The Game:

People are nodes who want to buy connections/vertices to other players. There needs to be a connection from each player to another. The cost is a function of perceived distance to others plus the cost to buy edges, which each player wants to minimize.

- n players
- α cost per edge
- Cost of player $i = \alpha(\text{number of edges they bought}) +$
(the sum of their perceived distances to every other player)
- Objective cost of solution is sum of all player's costs

Facts from last class: Optimal solutions, Nash Equilibriums, etc

For $\alpha \leq 2$, a complete graph is OPT

For $\alpha \geq 2$, a star graph is OPT

(Idea: If edges cost very little, just buy a bunch. If they cost a lot, buy as few as possible)

For $\alpha \leq 1$, a complete graph is a Nash equilibrium

For $\alpha \geq 1$, a star graph is Nash Equilibrium

This implied for $\alpha \geq 2$ and $\alpha \leq 1$, the price of stability = 1

For $\alpha \in (1, 2)$, price of stability $\leq \frac{4}{3}$

In most cases, we can match the best objective cost.

Price of Anarchy

What about the price of anarchy?

How bad can it get if we choose the worst equilibrium?

We will show that the price of anarchy (PoA) = $O(\sqrt{\alpha})$

(What is big O? Think of it as an asymptotic upper bound)

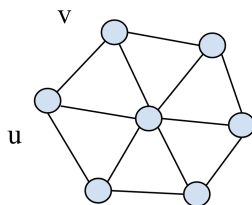
It is known that for $\alpha = O(\sqrt{n})$, the PoA = $O(1 + \frac{\alpha}{\sqrt{n}})$.

It is also known that for $\alpha \geq \Omega(n \log n)$, PoA = $O(1)$.

(What is omega? Think of it as an asymptotic lower bound)

The standard conjecture is that for any α , $\text{PoA} = O(1)$. **Not** proven!

Definition: Diameter



In the image above, the shortest path from u to v is 1.

The diameter is 2, as no two vertices are greater than 2 apart.

Plan for proving PoA

We'll be proving two claims:

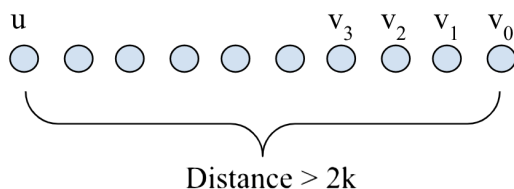
- **Claim 1:** If G is a Nash Equilibrium solution to this game, then the diameter of G is $2\sqrt{\alpha}$.
- **Claim 2:** If G is a Nash Equilibrium and has diameter d , then the cost is at most $O(d) \times \text{OPT}$

Both these claims bound something about Nash Equilibrium solutions.

If we know both these two claims, then $\text{PoA} = O(\sqrt{\alpha})$ is true because then $O(d) = O(2\sqrt{\alpha})$ and therefore cost at most (according to claim 2) is $\sqrt{\alpha} \times \text{OPT}$.

Proving claim 1

Idea: Look at vertices that are far apart, and if they're too far, then it's not a Nash Equilibria, and they'd prefer to buy an edge. We'll be looking at two vertices that are more than $2k$ apart and showing that the preference would be to buy the edge between them.



Say the distance between the two far apart vertices u and v is greater than $2k$.

From player u 's perspective:

If they bought the edge, their cost goes up by α . Also, if they bought the edge, then their perceived distance goes from $2k$ to 1. So, the path from u to v and the path v to u goes to 1. Also, before, the distance between u and every node between u and v incremented by 1.

We can look at the perceived distances to these with an added edge.

	Distance Before	Distance After
v_0	$> 2k$	1
v_1	$> 2k - 1$	2
v_2	$> 2k - 2$	3
v_{k-1}	$> 2k - (k - 1) = > k + 1$	k
v_k	$> k$	what it was before

In the table above, v_{k-1} is the last one that saves distance with an added edge.

So distance savings is at least:

$$(2k - 1) + (> 2k - 1 - 2) + (> 2k - 2 - 3) =$$

$$> (2k - 1) + (2k - 3) + (2k - 5) + \dots + 1 =$$

$$\sum_{i=1}^k (2k(2i - 1)) =$$

$$\sum_{i=1}^k 2k + \sum_{i=1}^k 1 - 2 \sum_{i=1}^k i =$$

$$2k^2 + k - 2 \frac{k(k + 1)}{2} =$$

(The fraction in the previous line comes from a formula)

$$2k^2 + k - k - k^2 = k^2$$

And we've now shown claim 1: if G is a Nash Equilibrium solution to this game, then the diameter of G is $2\sqrt{\alpha}$

Why we're done with claim 1:

A Nash Equilibrium means no one wants to change. In order to be a NE, the proposed buying of an edge has to be worse than what they have. Adding an edge gets a α increase to buy the edge. We're also gaining k^2 , as we've shown. So, for this strategy to be a NE, $k^2 < \alpha$ must be true for it not to be preferred to switch to it. So, $k < \sqrt{\alpha}$ and we are done.

This was the proof for the first claim that if G is a NE to this game, then the diameter of G is $\leq 2\sqrt{\alpha}$

Let's keep moving forward.

Proving Claim 2:

Our strategy:

- 1) Establish a “reasonable” lower bound for OPT
- 2) Establish a “reasonable” upper bound on the cost of the NE that is not “much more” than OPT

Starting with a lower bound for OPT:

What do we know about our cost? At the end of the day, we pay for all edges we buy and sum of all pairs of distances.

$$\text{OPT} = \alpha \times (\text{number of edges}) + \text{Sum of all paired distances}$$

How many pairs of distances are there? $n(n-1)$, roughly n^2

Can any of them be 0? No, So a lower bound of a distance is 0.

So, $\geq n(n-1)$ is a lower bound for the sum of all paired distance part of the equation.

There needs to be at least $n-1$ edges for all nodes to be connected (because of the depth first strategy for connecting every single player.)

So, the equation for the lower bound of the cost is:

$$\text{OPT} = \alpha \times (\text{number of edges}) + \text{sum of all paired distances}$$

$$\text{OPT} = \alpha(n-1) + n(n-1)$$

$$\text{OPT} = \Omega(\alpha n + n^2)$$

We have done the first part, established a “reasonable” lower bound for OPT.

Now, we will establish a “reasonable” upper bound on the cost of the NE that is not “much more” than OPT.

Our goal is to show that our cost is at most d times that of the OPT lower bound. Therefore, our goal is to show that our cost is:

$$O(\alpha dn + n^2 d)$$

Break our cost into “chunks”:

- 1) Perceived distances
- 2) Costs of buying edges

Our diameter is d . Every pair of vertices is at most d apart.

Looking at all perceived distances:

There are n players, meaning there are $n(n - 1)$ pairs of vertices. Each of these pairs is at most d apart. Therefore, the upper bound of perceived distances is dn^2 .

So for point **1**), all perceived distances are at most d at most $n(n - 1) = O(n^2)$ pairs, so $O(n^2 d)$

For point **2**) Cost of buying edges:

Break this part into two subsections:

- i) Edges that if I delete will disconnect the graph (in this case, perceived distance would jump to infinity)
- ii) Edges that don't

For point **i**):

How many can there be?

$$n - 1$$

Why $n - 1$? If I have more than $n - 1$ edges in a graph, somewhere I have a cycle. If I have a cycle, then one of those edges is not going to be something that disconnects everything.

Each of these edges costs α , so cost of these edges $\leq \alpha(n - 1) = O(n\alpha)$

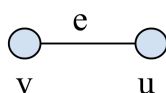
for **ii**) Edges that don't break apart the graph:

We can bound how many of these there are:

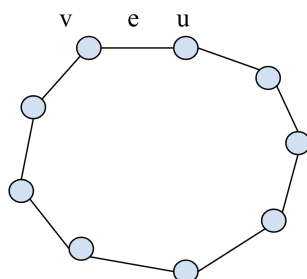
$$\leq 2dn^2$$

Nothing obvious about it, so let's figure it out. We'll look at a particular vertex, look at an edges that wouldn't disconnect the graph, and make some judgement calls.

Pick a vertex v and u connected by one of these edges e .



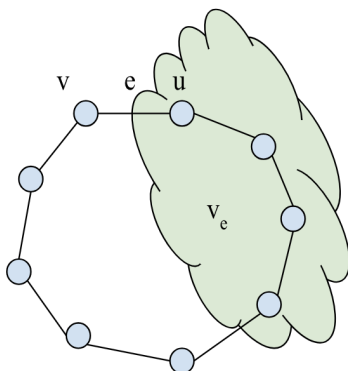
Deleting e doesn't change that v and u are still connected when looking at the entire graph.



Getting rid of e means the shortest path is to go all the way around.

Idea: look at the vertices that use edge e to get from v to that edge, that is e is on the shortest path to from v to that edge.

So, we're looking where the direction including e is smaller than the other direction, call those vertices v_e .



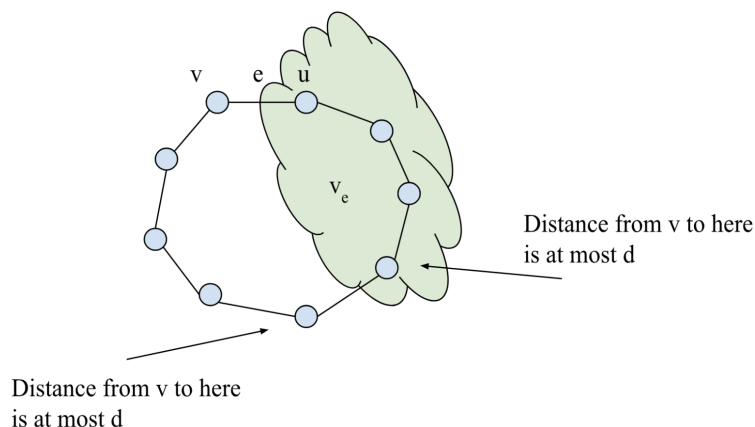
This is supposed to be a NE, meaning we don't want to delete that edge e . The cost α is keeping

this edge is smaller than all loss of not having the edge.

After deleting e , what is the distance between u and v , $\text{dist}(u, v)$?

Definitely at most n in the absolute worst case, but want a stricter bound in terms of d .

What is the distance in the old graph from v to last vertex in v_e ?



We can see the distance between v and the last vertex in v_e , and the distance between v and the last vertex not in v_e is both at most d , so:

$$\text{dist}(u, v) \leq 2d$$

In particular, $d(u, w) \leq 2d$ for any w in v_e .

So, if I were to lose e , what is an upper bound in terms of v_e of how much my cost would go down for v (v 's perceived distance to others)?

The only ones changing are those in v_e , and they decrease by at most $2d$. So, if $|v_e|$ is the number of things in v_e , then the upper bound for decrease in cost for v is $2d|v_e|$.

Because it's an NE, no one wants to deviate. The proposed strategy in this case is to get rid of e . Having e is NE, so we'd prefer to keep e than not have it. We have to pay α to have bought e . We would lose $2d|v_e|$, so $2d|v_e| \geq \alpha$.

We have shown that since this is a NE, $\alpha \leq 2d|v_e|$.

We want a specific bound on number of vertices that you can cut.

We know $\alpha \leq 2d|v_e|$, so $\frac{\alpha}{2d} \leq |v_e|$.

For every vertex in the graph, there are one or multiple shortest paths from v to that vertex. If greater than one, we can declare one of them as the chosen shortest path, and therefore we can say that each vertex should be exactly one v_e -like group. Therefore, the sum of $|v_e|$ over all edges leaving v should be at most n . Therefore:

$$\sum_{\text{non-cut edges leaving } v} \frac{\alpha}{2d} \leq n$$

Here, “cut” edges are those that would not disconnect the graph if they were removed.

We can rewrite this as:

$$\begin{aligned} (\text{cut edges leaving } v) \frac{\alpha}{2d} &\leq n \\ (\text{cut edges leaving } v) &\leq \frac{2dn}{\alpha} \end{aligned}$$

So, we have an upper bound on the number of cut edges. Therefore, we have an upper bound on the cost on one of these edges: $\alpha \frac{2dn}{\alpha} = 2dn$. We can sum this over all edges to get:

The cost of buying edges that do not disconnect the graph if deleted is $\leq 2dn^2$

Add this to the previous finding to find the upper bound for total cost of buying all edges of both kinds:

$$\text{Total cost} \leq O(n\alpha) + 2dn^2 = O(dn^2)$$

We can add this to the cost of perceived distances established in point **1**), $O(n^2d)$:

$$\text{Total cost} \leq O(dn^2) + O(n^2d) = O(2n^2d) = O(n^2d)$$

Because our upper bound is less than $O(adn + n^2d)$, d times the “reasonable” upper bound on the cost established earlier, we have established our second claim: If G is a Nash Equilibrium and has diameter d , then the cost is at most $O(d) \times \text{OPT}$.

Combining our first and second claims, we have shown that the price of anarchy is $O(\sqrt{\alpha})$.