

1 A Brief Review

The local connection game has n players, each of who want to get the shortest connection to every other pair of players. However, they must pay a cost α for every edge they construct. Notably, they can use any edge regardless of who paid for it originally. The cost function for each player is $\alpha \cdot \text{edges bought} + \sum_{\text{summed over all players}} \text{the number of edges to get to that player}$. Two completely unconnected players have a cost of ∞ . The objective function we use is the sum of all player's costs, which we try to minimize.

Last class we already derived several things about the optimal solutions and Nash equilibria for various α . If $\alpha \leq 1$ we want a complete graph for the optimal solution, which is a Nash equilibrium, and $\alpha \geq 2$ means that we want a star graph in the optimal solution, which is also a Nash equilibrium. Obviously in those cases the price of stability is 1, since the optimal and best Nash strategy are equivalent. When $\alpha \in (1, 2)$ we have the price of stability at most $\frac{4}{3}$, as a star graph is a Nash equilibrium but a complete graph is optimal. So, the ratio is at most the ratio in objective cost between a star and complete graph. The general intuition here is that the more generous complete graph wins with cheap edges and the stingier star graph wins with expensive edges. So for almost all α we at least have an optimal graph as a Nash equilibrium.

Figure 7.1 below shows what complete and star graphs look like.

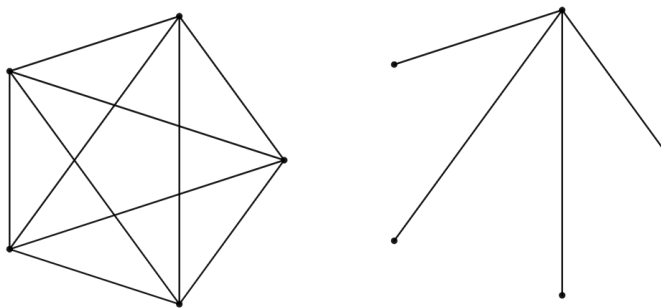


Figure 7.1: A complete graph with 5 points on the left, and a star graph with 5 points on the right (to see why it's a star, move the topmost point to the center).

2 How bad can it get? The price of anarchy

The remainder of class will be used to prove

Theorem 7.1. *The price of anarchy in the local connection game is $O(\sqrt{\alpha})$.*

Note that O denotes an asymptotic upper bound. We also know that for $\alpha \geq \Omega(n \log n)$ (here Ω is an asymptotic lower bound) that the price of anarchy is $O(1)$ and when $\alpha = O(\sqrt{n})$ that the price

of anarchy is $O(1 + \frac{\alpha}{\sqrt{n}}) = O(1)$ for arbitrarily large n . It is believed, but not proven, that the price of anarchy for any α is $O(1)$.

To show our theorem, we will prove two lemmas.

Lemma 7.2. *If G is a Nash equilibrium then the diameter of G is at most $2\sqrt{\alpha}$.*

Lemma 7.3. *A Nash equilibrium with diameter d has a cost at most $O(d)$ times the optimal strategy.*

Clearly, if both are true then G has a cost at most $O(\sqrt{\alpha})$ times the optimal solution, making the price of anarchy $O(\sqrt{\alpha})$. The diameter of a graph is the maximal length of the shortest path between two vertices. In Figure 7.1 this is 1 for the complete graph and 2 for the star graph.

2.1 Bounding the diameter of Nash equilibria

To prove 7.2 we'll be proving that for vertices more than $2k$ distance apart we would have preferred to buy the edge between them for some k , and then show that k is $\sqrt{\alpha}$. If this is true, then having a diameter more than $2k$ is not an equilibrium as the players will prefer to buy the edge connecting the distant points.

Let us take this graph of more than $2k$ diameter and extract the two points further apart and their shortest connecting path (for graphical ease I will say that $k = 4$). u is considering buying a direct route to v . See 7.2. The negative of this purchase is that it sets u α back. However, it gives u back more than $2k - 1$ in saved distance (as the original distance to v was $> 2k$ and now it is 1). Are there any other savings? There are, if we use more notation.



Figure 7.2: u and v , more than $2k$ apart. What should u do?

Call v_x the point on the connecting path x points away from v . This system is depicted in 7.3. The connecting path from u to v guarantees a path from u to v_x of $1 + x$ length (1 to get from u to v and x to get from v to v_x by definition). We also know that it takes more than $2k - x$ distance to get to v_x without the path: otherwise, we can get to v from u in $2x$ distance by going from u to v_x in less than or equal to $2k - x$ distance and from v_x to v in x distance. So, the distance savings using the newly-bought direct path to v from u would be more than $(2k - x) - (1 + x) = 2k - 2x - 1$. This is $2k - 1$ for $v_0 = v$, $2k - 3$ for v_1 , $2k - 5$ for v_2 , and so on until we hit v_{k-1} with a savings of at least 1 and then v_k with a savings of at least -1 . We can no longer guarantee the newly-bought path is shorter, so we see that we only save more than $2k - 1 + 2k - 3 + \dots + 1$ distance.

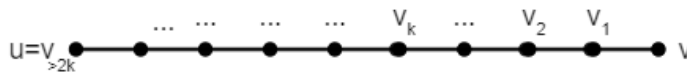


Figure 7.3: u and v , now labeled

Since the route costs α to build, we need that $2k - 1 + 2k - 3 + \dots + 1 = \alpha$ to know that the distance savings exceed α . This sum is equivalent to:

$$\begin{aligned}
\sum_{x=1}^k (2k - 2(x - 1) - 1) &= \sum_{i=1}^k 2k - \sum_{x=1}^k 2(x - 1) - \sum_{x=1}^k 1 \\
&= 2 \sum_{i=1}^k k - 2 \sum_{x=1}^k (x - 1) - \sum_{x=1}^k 1 \\
&= 2k^2 - 2 \frac{(k - 1)k}{2} - k \\
&= 2k^2 - (k^2 - k) - k \\
&= k^2
\end{aligned}$$

To have $2k - 1 + 2k - 3 + \dots + 1 = k^2 = \alpha$ and demonstrate that the extra edge would decrease u 's cost we need $k = \sqrt{\alpha}$. In that case, we see that the graph with more than $2k = 2\sqrt{\alpha}$ diameter can not be a Nash equilibrium as player u would prefer to buy an edge over the current state of affairs. We now move on to the second lemma.

2.2 Bounding the price of anarchy

To find an upper bound on the price of anarchy, we use a lower bound for the optimal cost and an upper bound for the worst Nash equilibrium. This will allow an upper bound for the price of anarchy $= \frac{\text{worst Nash equilibrium}}{\text{optimal cost}}$, as the denominator will be smaller or equal to than the actual optimal cost and the numerator will be larger or equal to the worst Nash equilibrium.

A lower bound for the optimal cost is the sum of a lower bound for the pairwise distances and the sum of a lower bound for the cost of the edges. We see that there are $n(n - 1) = n^2 - n$ pairwise distances in our objective function (each of the n player connects to $n - 1$ other players) with distance at least 1, since you must travel an edge to get to any other player. This is an $\Omega(n^2)$ pairwise distance cost.

For the cost of buying edges, we see that there needs to be at least $n - 1$ edges to connect every player using the depth-first strategy discussed previously for connecting every player. Even if we ignore that, we see that an edge connects at most 2 players so to connect n players we need at least $\frac{n}{2}$ edges. Either value is $\Omega(n)$, and each edge costs α , so we see that the cost is at least $\Omega(\alpha n)$ for the edges and at least $\Omega(n^2 + \alpha n)$ in total.

To show that the price of anarchy is at most $O(d)$ times the diameter, we need to show that $O(\alpha nd + n^2 d)$ is an upper bound for the cost of Nash equilibria. We will instead show that $O(\alpha n + n^2 d) \leq O(\alpha nd + n^2 d)$ is an upper bound.

2.2.1 Finding the upper bound of the cost of Nash equilibria

The cost is two parts: distances and total edge cost. We will first demonstrate that the sum of distances are $O(n^2 d)$. The diameter is at most d and all distances are at most the diameter. There are $O(n^2)$ distances. Therefore, the sum of distances is $O(n^2 d)$.

Next, we will prove that the sum of the costs of edges is bounded sufficiently tight. To see this, we see by the law of the excluded middle that all edges would either disconnect the graph (set some distances to ∞) or not disconnect the graph. The first case, the set of disconnecting edges, can only contain $n - 1$ edges. If there were more then we have a cycle for one of those edges, so we can break the cycle without making any part of the graph inaccessible from any other part of the graph. This makes the cost at most $\alpha(n - 1) = O(n\alpha)$ for disconnecting edges.

Non-disconnecting edges are trickier. We will show that there are at most $\frac{2dn^2}{\alpha}$ non-disconnecting edges. Essentially, we will show that a graph with any more edges would have added some edges that do not justify the α cost and would make the graph no longer Nash. So, we take the non-disconnecting edge e and disconnect it from its connection between v and u . There is, of course, still an edge connecting v and u .

Now look at the set V_e of all vertices that would use e to get to u , which is by the bidirectional nature of the game the vertices that u would use e to get to in the shortest manner. v is the obvious example here, but there may be others. Removing e does not impact the pairwise distance to get to u for any other points by definition, so the pairwise distance gained is only gained among the members of V_e . For all of V_e , how much distance does the loss of e add for the distance to u ? First we go from u to a vertex not in V_e but neighboring a vertex in V_e . This takes at most d distance, as it took d distance to get to that vertex before and removing e doesn't change the distances for any vertices not in V_e . Then, by being a neighbor, it takes 1 distance to enter V_e . From any point in V_e to v is at most $d - 1$ distance, as in the original graph we would use e to get to u , which is at most d distance by the diameter of the original graph. Going to u using e , as all vertices in V_e prefer, would first involve going to v so they could use e . This means that v must be at most $d - 1$ steps away. This is a total of $d - 1 + 1 + d = 2d$ distance for any point in V_e . This means that the distance added in total is at most $2d|V_e|$, as only the members in V_e added distance, which was at most $2d$ distance.

Since having e in the graph was a Nash equilibrium, we know that the distance saved by e , at most $2d|V_e|$, was more than the cost of α . So, $2d|V_e| \geq \alpha$ and therefore $|V_e| \geq \frac{\alpha}{2d}$. Since the $|V_e|$ from u for each edge e from u can be said as the vertices that u would use that edge to go to, and since the shortest path from u to any vertex uses only one edge, each vertex is in at most one $|V_e|$ from u . Therefore, $n \geq |V_e|\deg(u)$ where $\deg(u)$ is the number of edges exiting u , which means $|V_e| \leq \frac{n}{\deg(u)}$ and therefore $\alpha \leq 2d|V_e| \leq \frac{2dn}{\deg(u)}$, and therefore $\deg(v) \leq \frac{2dn}{\alpha}$. The total number of edges is at most the sum of the number of edges coming out of every point (since every point comes out of some edge), and there are n points, so the number of non-disconnecting edges is at most $n\deg(v) \leq \frac{2dn^2}{\alpha}$, and the cost is at most $\alpha \frac{2dn^2}{\alpha} = 2dn^2 = O(n^2d)$.

This means that there is at most $O(n\alpha) + O(n^2d) = O(\alpha n + n^2d)$ cost for any Nash equilibrium, which is at most $O(d)$ times the minimal cost of $\Omega(n^2 + \alpha n)$. We know $O(d) = O(2\sqrt{a}) = O(\sqrt{a})$ by 7.2, so we have proven that the cost of anarchy is at most $O(\sqrt{a})$, which demonstrates 7.1.