

CS381 Assignment 1 Solutions

Due September 18 2023 11:59 PM AoE

1. In class we worked on a problem that we labelled *Tragedy of the Commons*.

- (a) In class I claimed that there is no dominant strategy for any agent in this game. Explain what this means and show that it is true.

Solution: In this problem, we defined the utility of player i to be $x_i(1 - \sum_j x_j)$, which is uniquely maximized when $x_i = \frac{1 - \sum_{j \neq i} x_j}{2}$. In particular, this means that the best response for player i depends on the values chosen by the other players. For example, when $\sum_{j \neq i} x_j = \frac{1}{2}$, then i 's best response is to choose $x_i = \frac{1}{4}$. But if $\sum_{j \neq i} x_j = \frac{1}{4}$, then i 's best response is to choose $x_i = \frac{3}{8}$. As there is no single choice which is the best response under all possible settings of the other players, there is no dominant strategy for any player.

- (b) Suppose there are n players currently utilizing the strategies specified by the Nash Equilibrium determined in lecture. Suppose a new $n + 1$ st player arrives, who chooses their bandwidth x_{n+1} by calculating their best response assuming the first n players don't change, followed by player 1 rechoosing their bandwidth x_1 accordingly, followed by player 2, and so on until repeating as necessary until no player wants to unilaterally deviate from their strategy. Prove or show experimentally that these values converge to the solution determined in lecture, ie $\lim x_i = \frac{1}{n+2}$ for all i . If you solve this experimentally, you can use $n = 10$ and I would expect to see plots of x_1 's and x_{n+1} 's values over time (no need to submit code).

- (c) In class Thor suggested that the reason I had labelled this game a *Tragedy of the Commons* was that each player could choose $x_i = \frac{1}{2n}$ giving each player a value $\frac{1}{2n}(1 - n \frac{1}{2n}) = \frac{1}{2n} \frac{1}{2} = \frac{1}{4n}$, which is clearly better than $\frac{1}{(n+1)^2}$ for $n > 4$. First, confirm that this solution is not a Nash Equilibrium.

Solution: The best response from any player i in this situation is given by $x_i = \frac{1 - \sum_{j \neq i} x_j}{2} = \frac{1 - \frac{n-1}{2n}}{2} = \frac{n+1}{4n} \neq \frac{1}{2n}$, so player i wants to deviate, which means this is not a Nash Equilibrium.

- (d) Show that Thor's value wasn't accidental - this is explicitly the specification of strategies that gives the best total value. If you'd like, you can assume I'm only talking about solutions where everyone chooses the same proportion of the bandwidth - for some amount of extra credit, ignore this assumption.

Solution: The total value is given by $\sum_i x_i(1 - \sum_j x_j)$. Letting $T = \sum_i x_i$, this gives us that the total value is $T(1 - T)$, which is maximized when $T = \frac{1}{2}$. Hence any solution which gives $T = \frac{1}{2}$ gives maximal total value: Thor's setting is such a solution.

2. So far in class we've discussed "identifying" Nash Equilibria but haven't discussed actually finding them algorithmically.

- (a) Suppose someone gives you the payoff matrices for a two player game (this is called *normal form*) between players P_1 and P_2 : specifically matrices A and B are both $m \times n$ matrices where A denotes the value for P_1 for each pair of strategies and B denotes the values for P_2 . Describe an $O(m \times n \times (m + n))$ time algorithm for finding all *Pure* (deterministic) Nash Equilibria. As we shall see in lecture, finding all *mixed* Nash Equilibria is far more difficult.

Solution: Consider the following pseudocode:

```
findPureEquilibria(A, B):
    Initialize Out as an empty set
    for each row r
        for each column c
            confirm that A[r][c] is the maximal entry in its column
            confirm that B[r][c] is the maximal entry in its row
            if both are confirmed, then add (r, c) to Out
    return out
```

This algorithm iterates over every possible specification of pure strategies (of which there are $m \times n$) and then checks if each one is a Nash Equilibrium, which is equivalent to making sure that no player wants to deviate, which is achieved by the two "confirmation" steps. Note that implementing these steps take $O(n + m)$ time, giving an $O(m \times n \times (m + n))$ time algorithm.

- (b) In class we looked at the Prisoners' Dilemma problem, with payoff matrix below:

	-10	-20
-10	0	-1
-20	-1	-1

In the above matrix, the bottom left of each cell is the payoff to P_1 , whereas the top right is the payoff to P_2 . In class Henry pointed out that the strategy of tattling on the other was *dominant* - for example, the top row payoffs are strictly better for P_1 than their corresponding payoffs in the same column in the second row. Hence the first row strategy *dominates* the second row strategy.

Consider the following algorithm: successively delete dominated strategies by each player. If the matrix is ever reduced to a single cell, ie a single strategy for each player, declare that strategy a Nash Equilibrium.

First, observe that this algorithm works for the Prisoners' Dilemma above. Second, observe that if a game lacks dominant strategies, then this algorithm won't be too useful. Lastly, decide whether this algorithm works in general: that is, if it reduces the game to a single cell, is that cell necessarily a Nash Equilibrium **in the original game**? If it is, explain why, otherwise give a specific game counter example.

Solution: Such a cell c would necessarily be a Nash Equilibrium: suppose it weren't. Then at least one player would prefer to deviate - let's assume it's the row player. Then there is another cell c' in the same column with a better payoff for the row player - specifically, look at cell which was deleted last from this column. This cell c' was at some point deleted as it was dominated by another row r'' (it couldn't have been dominated

by another column, as it is in the same column that the final cell is in). The cell c'' in this row r'' and in the same column as c' has either been deleted later (which would be a contradiction by choice of c') or is the cell c itself, in which case the row player isn't actually deviating.

- (c) Consider the following algorithm for a 2-player game given in normal form: pick any starting cell. Alternating between players, each player moves the currently selected cell by selecting the cell corresponding to the *best response* assuming the other player plays the strategy according to the currently selected cell. Determine a normal-form representation (just a payoff matrix) of a game where this algorithm will not terminate despite there being a Nash Equilibrium (which shows that the strategy we used in 1b need not actually converge, generally). **Hint: I am pretty sure you need at least 3 rows and columns for this. My solution has 3 strategies for each player.**

Solution:

	-1	1	0
1	-1	0	0
	1	-1	0
-1	1	0	0
	0	0	10
0	0	10	