

CS381 Assignment 3 Solutions

Due October 19 2023 11:59 PM AoE

1. Recall the *Local Connection Game* we studied in class: in this game, the n players are modeled as vertices who have the option of buying edges to other players/vertices at the cost of α per edge. The perceived resultant cost c_i of player i is the summed costs of their edges they paid for and their perceived distances to every other vertex (if a vertex cannot be reached, their perceived distance is infinite). Specifically,

$$c_i = \alpha E_i + \sum_{j \neq i} \text{dist}(v_i, v_j),$$

where E_i is the number of edges purchased by player i and v_x denotes the vertex corresponding to player x . We defined the social cost c_G of a graph G to be

$$\sum_i c_i = \alpha E + \sum_{\text{all pairs } (i,j)} \text{dist}(v_i, v_j),$$

where E is the total number of edges purchased by all players. For n players, let G_n^* denote the graph that minimizes the social cost function (in class, we showed that if $\alpha \leq 2$ then G_n^* is a complete graph, and if $\alpha \geq 2$ then G_n^* is a star graph).

In class, we established the following two lemmas:

Lemma 1. For a graph G produced by n players in a state of Nash Equilibrium, we have $\frac{c_G}{c_{G_n^*}} \leq d_G$ where d_G is the diameter of G .

Lemma 2. The diameter d_G of a graph G produced by players in a state of Nash Equilibrium satisfies $d_G = O(\sqrt{\alpha})$.

Together, these two lemmas immediately gave us the theorem:

Theorem 1. The Price of **Anarchy** of the Local Connection Game is $O(\sqrt{\alpha})$.

Our goal in this question will be to improve this a bit.

- (a) First, notice that if α grows unboundedly, the bound above for the Price of **Anarchy** is also unbounded. Let's fix that; specifically, let's show that for $\alpha > n^2$, we have that the Price of **Anarchy** is $O(1)$.

- i. If $\alpha > n^2$, show that any **connected** graph with a cycle must not be a Nash Equilibrium.

Solution: Suppose G is a graph produced containing a cycle. Consider any player paying for an edge in the cycle: they could stop paying for it with a savings of α , but

their distances could go up. Specifically, since this edge is part of a cycle, the graph would not become disconnected, so the distance to every other vertex could increase by at most $n - 1$. As there are $n - 1$ other vertices, this means the distance part of their cost function would increase by at most $(n - 1)^2$.

- ii. **Extra Credit: (Shout out to Clio):** Even if $\alpha > n^2$, not every graph without a cycle is a Nash Equilibrium.
- iii. If $\alpha > n^2$, show that the price of **Anarchy** is $O(1)$. Hints etc: You can assume $n \geq 2$, which means $\alpha > 4$, so the optimal solution is a star; the game is pretty dull with $n = 1$.

Solution: We know that for $\alpha > 1$, the optimal solution is a star graph. Hence, the price of anarchy for $\alpha > n^2$ is just the maximal ratio of the social cost of a tree to a star.

The cost of the optimal solution, as we computed in class, is the cost of the $n - 1$ edges plus the perceived distances of every vertex to every other vertex. In the case of the star graph, this is $\alpha(n - 1) + 2(n - 1) + 2(n - 1)(n - 2) = (n - 1)(\alpha + 2n - 2)$. To bound the cost of a tree, we note that every vertex is at most distance $n - 1$ from any other vertex. Hence, the social cost of a tree is at most $\alpha(n - 1) + n(n - 1)(n - 1) = (n - 1)(\alpha + n(n - 1))$. Hence, the PoA for $\alpha > n^2$ is at most

$$\begin{aligned} & \frac{(n - 1)(\alpha + n(n - 1))}{(n - 1)(\alpha + 2n - 2)} \\ &= \frac{\alpha + n(n - 1)}{\alpha + 2n - 2} \\ &= \frac{\alpha}{\alpha + 2n - 2} + \frac{n(n - 1)}{\alpha + 2n - 2} \\ &\leq 1 + 1 \end{aligned}$$

where the first fraction is at most 1 as $2n - 2 \geq 0$ and the second fraction is at most 1 as $\alpha > n^2 > n(n - 1)$.

Do either b or c below or both for some amount of extra credit and my adoration (I think b is actually much easier, and if your initials are ZG and are still interested in AIT you must do b):

- (b) Let's tighten up Lemma 2 above: specifically, let's show that $d_G = O(1 + \frac{\alpha}{\sqrt{n}})$. Throughout this problem, we'll be assuming G is a graph produced by n players in a state of Nash Equilibrium. We already know that for small diameter Nash Equilibria, the price of **Anarchy** is effectively constant (by Lemma 1 above), so let's try and deal with bigger diameters.

Let $d' = \lfloor \frac{d-1}{4} \rfloor$, ie the result returned by Java's integer division when I divide $d - 1$ by 4, that is the value received by calculating $\frac{d-1}{4}$ and deleting the non-integer part. Let u and v be vertices such that $\text{dist}(u, v) = d$. Let B be the collection of vertices at most d' distance from u . We want to investigate the choice of players u and v to not buy more edges to save themselves cost in perceived distance. Let w be any vertex in B .

- i. Under the assumption that d is reasonably big, confirm $\alpha > d > d'$. (just use Lemma 2 above)

Solution: In class, we showed that $d < \sqrt{\alpha}2 < \alpha$, and clearly $d' < d$.

- ii. Confirm that $\text{dist}(w, v) \geq d - d'$.

Solution: Note that $\text{dist}(u, v) \leq \text{dist}(u, w) + \text{dist}(w, v)$, where $\text{dist}(u, w) \leq d'$ by

construction of B . So if $\text{dist}(w, v) < d - d'$, then $\text{dist}(u, v) < d' + d - d' = d$, which contradicts the specific choice of u and v .

- iii. If v were to buy the edge to player u , confirm that $\text{dist}(w, v) \leq d' + 1$.

Solution: If this edge existed, then one could get from w to v by going through u . As above, we know that $\text{dist}(w, u) \leq d'$, so with the addition of the edge from u to v we would be able to get from w to v with distance at most $d' + 1$.

- iv. Confirm that $|B|^{\frac{d-1}{2}} \leq \alpha$ where $|B|$ is the size of the collection B .

Solution: We know that the graph is currently at Nash Equilibrium, so v doesn't want to purchase the edge to u . This means that α is greater than the decrease in distance v would see to other vertices. We know that $\text{dist}(w, v) \geq d - d'$ without the edge from v to u , and would be at most $d' + 1$ with the edge. Hence, the distance to every vertex in B would decrease by at least $d - d' - (d' + 1) = d - 2d' - 1 \geq d - 2\frac{d-1}{4} - 1 = \frac{2d-2}{4} = \frac{d-1}{2}$. Hence, the cost of α is greater than the distance savings, which is at least $|B|^{\frac{d-1}{2}}$, as desired.

- v. Suppose $\text{dist}(u, w) = d'$ and let A_w be the collection of vertices x for which the shortest path from u to x goes through w and immediately exits B (to a vertex that is distance $d' + 1$ from u).

Confirm that if u bought the edge to w they would reduce their costs due to perceived distances by at least $|A_w|(d' - 1)$, where $|A_w|$ is the number of items in A_w .

Solution: Since the shortest path currently goes through w , if u bought the edge to w then the shortest path from u to any $x \in A_w$ would take the new edge from u to w , which shortens the path by $d' - 1$. As such, their distance costs would decrease by at least $|A_w|(d' - 1)$.

- vi. Confirm that $|A_w|(d' - 1) \leq \alpha$.

Solution: We know this graph is currently at Nash Equilibrium, so u doesn't want to purchase the edge to any w , which means the cost α is more than the distance savings, which we know to be at least $|A_w|(d' - 1)$.

- vii. Confirm that there must be a vertex w in B where $|A_w| \geq \frac{n-|B|}{|B|}$. Hint: Pigeon Hole Principle

Solution: Outside of B , there are $n - |B|$ vertices to distribute among the unique A_w s, of which there is just one set per vertex in $|B|$. As such, there must be a w such that $|A_w| \geq \frac{n-|B|}{|B|}$.

- viii. Combine your above inequalities to get $\alpha^2 \geq n(d' - 1)^2$ (probably the most annoying step, pretty much just comparing big and small things above).

Solution: By combining vi and vii, we get $\frac{n-|B|}{|B|} \leq \frac{\alpha}{d'-1}$, which we can rearrange to get $n \leq (\frac{\alpha}{d'-1} + 1)|B| \implies n \leq 2\frac{\alpha}{d'-1}|B|$ (as $\frac{\alpha}{d'-1} > 1$). Combining with iv, we get $\frac{n(d'-1)}{2\alpha} \leq \frac{2\alpha}{d-1}$. Rearranging this inequality, we get $\alpha^2 \geq \frac{1}{4}n(d'-1)(d-1) \geq n(d'-1)^2$.

- ix. Show that the Price of **Anarchy** is $O(1 + \frac{\alpha}{\sqrt{n}})$.

Solution: Noting that $d = O(d')$ and rearranging viii to get that $d' = O(1 + \frac{\alpha}{\sqrt{n}})$, we get that $d = O(1 + \frac{\alpha}{\sqrt{n}})$.

- (c) This question investigates the rate of convergence for the local congestion game on random graphs where everyone uses best responses to determine whether they buy any new edges or give up any currently owned edges.

- i. Create a program that creates random graphs of n vertices according to the following process: given some probability p , for each (unordered) pair of vertices u and v ,

- create an edge between them with probability p . If an edge is created, set u to be the 'buyer' with 50% probability and otherwise set v to be the owner.
- ii. Loop through the players cyclically and have each player play their best response to the current state until the graph is in a state of Nash Equilibrium. Record the number of iterations (1 iteration = 1 player's turn) it took to reach equilibrium.
 - iii. Letting $n = 40$, as a function of p and α analyze how many iterations on average it takes to reach equilibrium. I would expect a significantly large number of trials to be run to create data to back up whatever claims you make in your analysis.
- (d) **Extra Credit:** Show that the Price of Anarchy in this game is $O(1)$ regardless of α . Hint: Have an insight that every person who has ever looked at this problem so far has not had.
2. Recall the Shapley Network Design game from lecture: in this problem, we are given a directed graph G where each edge e has an associated cost c_e . Each player i has an associated pair of vertices s_i and t_i and must choose a path in the graph from s_i to t_i . The cost c_i player i receives is their share of the edges they use in their path P_i :

$$c_i = \sum_{e \text{ in } P_i} \frac{c_e}{k_e}$$

where k_e is the number of players whose path includes traversing e . The social cost of a solution C_S (S can be thought of as a specification of a collection of edges that connects all players s/t pairs) is

$$C_S = \sum_{e \text{ in } S} c_e = \sum_{\text{players } i} c_i$$

- (a) In class we showed that the Price of Anarchy of this game is at least n , where n is the number of players. In fact, it can never exceed n .
- i. Let P_i^* be the minimal cost path from s_i to t_i if this were a 1 player game for just player i (equivalently, P_i^* is the minimal cost path from s_i to t_i if costs were not shared by players). Confirm that in a Nash Equilibrium state,

$$c_i \leq \sum_{e \text{ in } P_i^*} c_e.$$

Solution: Note that the cost of a particular path for a player is at most the sum of the costs of the edges, due to the cost sharing implementation. Hence, the cost of the path a player chooses in a Nash Equilibrium state must be better than the cost of any other path, in particular the minimal path.

- ii. Confirm that

$$\sum_{e \text{ in } P_i^*} c_e \leq C_{\text{OPT}}$$

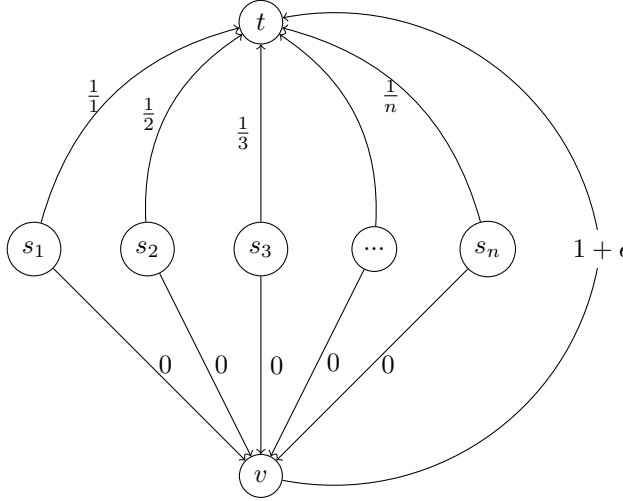
where OPT is the minimal cost solution.

Solution: The objective cost of a solution is just the sum of the costs of the edges used. There must be a path from s_i to t_i , so the cost of that path must be at least the sum of costs of the edges in P_i^* .

- iii. Confirm that the Price of Anarchy of this game is at most n . Hint: take the above and sum over all the players.

Solution: By combining i and ii, we know $c_i \leq C_{\text{OPT}}$, so $C_S = \sum_i c_i \leq \sum_i C_{\text{OPT}} = nC_{\text{OPT}}$, which shows that $\frac{C_S}{C_{\text{OPT}}} \leq n$ for any Nash Equilibrium solution S .

- (b) In class we showed that the price of stability is at most $H_n = \sum_{i=1}^n \frac{1}{i}$. We'll now show that the price of stability is at least H_n . Let $\epsilon > 0$ (imagine it's infinitesimally small though) and consider the following graph:



In the above graph, $t = t_1 = t_2 = \dots = t_n$, so every player has the same target.

- i. Confirm that there is exactly one (pure) Nash Equilibrium in this graph.

Solution: Suppose in some solution $k > 0$ players are taking the path through v : then some player $i \geq k$ is taking the path $s_i \rightarrow v \rightarrow t$. This path has cost $\frac{1+\epsilon}{k}$, which has cost greater than $\frac{1}{k}$. Hence they would prefer to deviate by just taking the direct edge $s_i \rightarrow t$. Hence the only solution which is a Nash Equilibrium is when $k = 0$, or more explicitly when every player i is taking the path $s_i \rightarrow t$.

- ii. Confirm that the Price of Stability of this game is H_n .

Solution: The minimal cost solution is when all players take the path $s_i \rightarrow v \rightarrow t$, giving a solution cost of $1 + \epsilon$. As the cost of the Nash Equilibrium in i is H_n , this gives a ratio of $\frac{H_n}{1+\epsilon}$, which can be made arbitrarily close to H_n .

- iii. Confirm that if this graph's edges were replaced with undirected edges, the price of stability of this specific graph would be 1.

Solution: In this case, the only Nash Equilibrium is the path $s_i \rightarrow v \rightarrow s_n \rightarrow t$, which has cost $\frac{1}{n}$, which matches the optimal solution.

- iv. **Extra Credit:** It is known that the Price of Stability for this game when the graphs are undirected is $O(\frac{\log n}{\log \log n})$, but a matching lower bound has not been found. Establish that the Price of Stability is $O(1)$ or $\omega(1)$ (ω is like Ω , except that it is *strictly greater*). So $\omega(1)$ means that the Price of Stability would be strictly greater than constant: that is, for any constant c given, you can find an instance where the Price of Stability on that instance is greater than c). Hint: Have an insight that every person who has ever looked at this problem so far has not had.