

CS375 Small Assignment 2 Solutions

Due Sep 12th 10PM ET

In class, we gave two (technically different) definitions of O : we say $f(n) = O(g(n))$ if

1. there exists some constant $c \geq 0$ such that $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} \leq c$.
2. there exists some constants $c, n_0 \geq 0$ such that $f(n) \leq c * g(n)$ whenever $n \geq n_0$.

We defined the terms Ω and Θ accordingly:

1. $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$
2. $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

For the purpose of this assignment (I doubt I'll use it again, but perhaps they might show up later), here are two more asymptotic notations:

1. $f(n) = o(g(n))$ if $f(n) \neq \Omega(g(n))$
2. $f(n) = \omega(g(n))$ if $f(n) \neq O(g(n))$

In class, we said O and Ω are like $\leq$ and $\geq$ respectively; in much the same way, o and ω are akin to $<$ and $>$ respectively. In the limit sense of things, $f(n) = o(g(n))$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$ (for ω , this limit would be ∞).

The main reason to get comfortable with these definitions is simply to mentally organize what the different 'classes' of runtime are, eg linear, quadratic, etc. For this homework (which is taken from CLRS problem 3.2), fill in the table with Xs where $f(n)$ is asymptotically bounded using the associated notation by $g(n)$. Assume that $c > 1$, $k \geq 1$ and $\epsilon > 0$. (To make it easier, you can assume k is an integer.)

$f(n)$	$g(n)$	O	o	Ω	ω	Θ
$\log^k(n)$	n^ϵ	X	X			
n^k	c^n	X	X			
$\sqrt{n}$	$n^{\sin(n)}$					
2^n	$2^{\frac{n}{2}}$			X	X	
$n^{\log(c)}$	$c^{\log(n)}$	X		X		X

No work shown necessary for these ones - however, I'll expect that you would know how to do them, especially since some of these will show up in some way on the upcoming problem set.

Solutions: I'll include some quick calculations from each of these.

For the first:

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{\log^k(n)}{n^\epsilon} \\
&= \lim_{n \rightarrow \infty} k \frac{\log^{k-1}(n) \frac{1}{n}}{\epsilon n^{\epsilon-1}} \\
&= \frac{k}{\epsilon} \lim_{n \rightarrow \infty} \frac{\log^{k-1}(n)}{n^\epsilon} \\
&= \dots \\
&= \frac{k!}{\epsilon^k} \lim_{n \rightarrow \infty} \frac{1}{n^\epsilon} = \infty
\end{aligned}$$

for $\epsilon \neq 1$. Obviously this changes a little bit with $\epsilon = 1$, but it's just easier.

Regarding n^k vs c^n , note that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{n^k}{c^n} \\
&= \lim_{n \rightarrow \infty} \frac{k n^{k-1}}{c^n \log(c)} \\
&= \frac{k}{\log c} \lim_{n \rightarrow \infty} \frac{n^{k-1}}{c^n} \\
&= \dots \\
&= \frac{k!}{(\log c)^k} \lim_{n \rightarrow \infty} \frac{1}{c^n} = 0
\end{aligned}$$

Because of the way $n^{\sin n}$ grows, there's just no way to characterize its growth via our standard polynomial/exponential/etc. functions.

Note that $2^{\frac{n}{2}} = 2^{n^{\frac{1}{2}}} = (2^n)^{\frac{1}{2}} = \sqrt{2^n}$, so comparing 2^n and $2^{\frac{n}{2}}$ is in some sense similar to comparing n and $\sqrt{n}$: clearly 2^n is asymptotically greater than $\sqrt{2^n}$.

Lastly, note that

$$\begin{aligned}
n^{\log c} &= (c^{\log_c n})^{\log c} \\
&= c^{(\log_c n)(\log c)} \\
&= c^{\log(c^{\log_c n})} \\
&= c^{\log n}
\end{aligned}$$

so the last two functions are exactly equal.