

CS386 Assignment 2

Due: Friday, October 2, 10 PM ET

A few protocols that we went over in class:

Cut-and-Choose

Number of Players Allowed: $n = 2$

Step 1: P_1 cuts the cake into two pieces.

Step 2: P_2 chooses one of the two pieces.

Step 3: P_1 receives the remaining piece.

Austin's Moving Knives

Number of Players Allowed: $n = 2$

Step 1: A knife is held above the cake and continually dragged from the left to the right until any player (say P_1) says “stop”, at which point P_1 takes hold of the knife exactly where it was when “stop” was called.

Step 2: P_1 then is also given a second knife which starts from the left edge of the cake. P_1 then moves both knives continuously (but not necessarily at the same rate) until either P_2 says “stop” or the first knife reaches the right edge of the cake, at which point the two knives are lowered to cut the cake exactly where they are currently held. I did some thinking about this - there's an additional requirement hidden here. As P_1 continuously drags the first knife to the right, they can do whatever they want with the other knife, conditioned on (a) its movement is continuous, and (b) at the exact moment the first knife would reach the right edge, the second knife must reach exactly where the first knife was when “stop” was called.

Step 3: When either of the above conditions are met, P_2 chooses either the slice created between the knives, or the piece(s) outside the knives.

Step 4: P_1 receives whatever is left.

Last Diminisher

Number of Players Allowed: $n \geq 1$

Step 1: If $n = 1$, give them the whole cake (and end the protocol).

Step 2: Each player makes a mark on the cake.

Step 3: Whoever made the mark furthest to the left makes a cut at their mark and receives the piece to the left.

Step 4: Recurse with all remaining players on what is left of the cake.

Dubins–Spanier

Number of Players Allowed: $n \geq 1$

Step 1: If $n = 1$, give them the whole cake (and end the protocol).

Step 2: A knife is held above the cake and moved gradually and continuously from the left to the right until any player (say P_1) calls “stop”.

Step 3: A cut is made where the knife currently is - the slice to the left of the knife is given to P_1 .

Step 4: Recurse with all remaining players on what is left of the cake.

Selfridge–Conway

Number of Players Allowed: $n = 3$

Step 1: P_1 cuts the cake into 3 pieces.

Step 2: P_2 labels the pieces S_1 , S_2 , and S_3 . P_2 may further subdivide S_1 into two pieces which they label S'_1 and R . If they choose not to subdivide, then $S'_1 = S_1$ and $R = \emptyset$.

Step 3: P_3 , P_2 , and P_1 (in this order) each choose a piece from $\{S'_1, S_2, S_3\}$. If P_3 doesn't take S'_1 , P_2 must.

Step 4: If $R \neq \emptyset$, then relabel P_2 and P_3 as P_A and P_B where P_A is the player who took S'_1 . Have P_B cut R into 3 pieces. P_A , P_1 , and P_B (in this order) each choose a piece from R .

1. In class we kinda hand-waved the proof for the exactness of Austin's Moving Knives protocol. Recall that a protocol is *exact* if for every player there is a strategy they can follow by which they can guarantee themselves **exactly** $\frac{1}{n}$ in value.
 - (a) Show that the protocol is exact. You will need both continuity and the intermediate value theorem (see? Calculus does matter).
 - (b) Show that without the red portion, the protocol is not exact.

2. For two of the above protocols other than Cut-and-Choose, show that they are not Pareto-Optimal. Recall that a protocol is Pareto-Optimal if *every* allocation produced by the protocol is Pareto-Optimal.

3. Manipulation and Honest Play.

- (a) Later on in this class we will discuss the idea of *incentive compatibility*. A mechanism is said to be *incentive compatible* if each agent interacting with it always achieves their best outcome by ‘playing honestly’ (reporting their true preferences). Show that Cut-and-Choose is **not** incentive compatible (here, we’ll assume ‘playing honestly’ means that the first player cuts the cake in half according to their valuations and the second player chooses their more preferred piece).

In fact, similar arguments can be made for all the above protocols. As a result, it’s reasonable to think of a different type of definition for manipulation: a cake-cutting protocol is **immune to manipulation for risk-averse players** if each player maximizes their payoff in the worst-case by ‘playing honestly’. For us, we’ll assume that ‘playing honestly’ means following a specific suggested strategy σ_i . Let $X_i(\tau = (\tau_1, \dots, \tau_n))$ denote the allocation player i receives when each player j follows strategy τ_j . Formally, we say a protocol is immune to manipulation for risk-averse players if:

$$(\forall i)(\forall \tau_i)(\exists \tau_{-i}) [\tau_i \neq \sigma_i \rightarrow v_i(X_i(\tau' = (\tau_1, \dots, \tau_{i-1}, \sigma_i, \tau_{i+1}, \dots, \tau_n))) > v_i(X_i(\tau))].$$

That is, for every player and every other strategy they might try other than the recommended σ_i , there is specification of strategies over the other players such that results in a worse payoff for i than if they had just played σ_i . There is a **weaker** notion replacing the strict inequality with a weak inequality:

$$(\forall i)(\forall \tau_i)(\exists \tau_{-i}) [\tau_i \neq \sigma_i \rightarrow v_i(X_i(\tau' = (\tau_1, \dots, \tau_{i-1}, \sigma_i, \tau_{i+1}, \dots, \tau_n))) \geq v_i(X_i(\tau))].$$

In this case, this bad strategy for them isn’t necessarily worse, but it is saying that for every alternate strategy for i other σ_i there is a specification of strategies they could play against where they would do no better than by playing the recommended strategy.

- (a) Prove that the **Cut & Choose** protocol with the proposed ‘honest’ strategy is immune to manipulation for risk-averse players. (You’ve essentially already argued this informally in lecture. The job here is to formalize it against the actual definition)
- (b) Show that Austin’s Moving Knives protocol with the proposed ‘honest’ strategy from class is not immune to manipulation for risk-averse players, but is immune under the weaker notion.
- (c) Of the other protocols we’ve covered (Last Diminisher, Dubins-Spanier, Selfridge-Conway) pick one and decide if it is immune to manipulation for risk-averse players for the ‘honest’ strategy as we defined it to be in class. Prove it.

4. A New Protocol: Even–Paz.

Even-Paz

Number of Players Allowed: $n \geq 1$

Step 1: If $n = 1$, give them the whole cake (and end the protocol).

Step 2: Have each player make a mark. Make a cut at the median of the marks (if an even number of marks, make a cut between the two middle marks)

Step 3: Assign everyone whose mark was to the left side of this cut and as many as needed from those who marked directly on the cut to get to $\lfloor \frac{n}{2} \rfloor$ players to the left piece and everyone else (the remaining $\lceil \frac{n}{2} \rceil$ players) to the right.

Step 4: Recurse on the two halves with the assigned participants to each half.

Show that Even-Paz is

- proportional
- not envy-free
- finite bounded
- immune to manipulation for risk-averse players
- not Pareto-Optimal

5. **The Axioms** Recall the four properties every valuation function v_i is assumed to satisfy: **Normalization** ($v_i([0, 1]) = 1$), **Positivity** ($v_i(A) > 0$ for all nonempty A), **Additivity** ($v_i(A \cup B) = v_i(A) + v_i(B)$ for disjoint A, B), and **Divisibility** (for all A and $\alpha \in [0, 1]$, there is a $B \subseteq A$ with $v_i(B) = \alpha \cdot v_i(A)$).

For each axiom other than Normalization and Positivity, construct a valuation profile satisfying the *other three* axioms but violating this one, under which **no proportional division exists**. That is, every possible division leaves at least one player with less than $\frac{1}{n}$ of their own total value.

6. **Finishing Stromquist's Impossibility Proof.** This section picks up exactly where Thursday's lecture left off, working from Stromquist's 2008 paper. Recall a **rigid measure system (RMS)** with parameters x, y, s, t ($0 < x < y < 1$, $0 < s < 1/3 < t < 1/2$, $s + 2t = 1$) is a set of three measures v_A, v_B, v_C with densities always strictly between $\sqrt{2}/2$ and $\sqrt{2}$, and with

	$[0, x]$	$[x, y]$	$[y, 1]$
v_A	t	t	s
v_B	s	t	t
v_C	t	s	t

- (a) In class we claimed that *every* envy-free division of an RMS has its cuts at exactly x and y . In lecture we handled two of the cases directly (cuts strictly inside $[x, y]$, and cuts strictly outside $[x, y]$). The remaining case is $\hat{x} \geq x$ and $\hat{y} \geq y$, with strict inequality in at least one (there's technically another case, but it's just symmetric to this one basically, so whatever). Let's call $S_1 = [x, \hat{x}]$ and $S_2 = [y, \hat{y}]$.

Show that

- Both A and C prefer $[0, \hat{x}]$ to the rightmost piece ($[\hat{y}, 1]$).

- ii. Hence, the rightmost piece must be given to B . Show that this implies $v_B(S_1) \geq 2v_B(S_2)$. (Start from the fact that B must prefer the right piece to the middle piece)
 - iii. Show that if $v_i(S) \geq 2v_i(T)$ for some pieces S and T , then $2v_j(S) > v_j(T)$ (where i and j are assumed to be different people).
 - iv. Combine the last two to show that both A and C prefer the left piece over the middle piece.
 - v. Put it together: explain why this means this proposed cut cannot produce an envy-free allocation.
- (b) The game we played was that every time a player managed to correctly mark x or y , we'd update them by choosing a new value $\hat{t}$ for t , then updating all the relevant parameters for the RMS. Determine *every* constraint this puts on our new value of t , and confirm that each constraint allows some little bit of wiggle room - enough that there is some new value of t we can choose that satisfies all the constraints while also changing x and y .