CS386 Assignment 7

Due: Friday 11/13 10 PM ET

Throughout, n voters $N = \{1, \dots, n\}$ rank a set A of m alternatives. A *profile* $\succ = (\succ_1, \dots, \succ_n)$ is a list of strict rankings, and $a \succ_i b$ means that voter i ranks a above b. Write $\text{top}(\succ_i)$ for the top-ranked alternative of voter i. The *majority relation* of $\succ$ is defined by $a \succ_{\text{maj}} b$ if strictly more than $n/2$ voters rank a above b. A *Condorcet winner* is an alternative c with $c \succ_{\text{maj}} o$ for every $o \neq c$; it is unique when it exists. A *voting rule* on a domain D (a set of profiles) is a function $f : D \to A$. It is *incentive compatible* (IC) if there are no profiles $\succ, \succ' \in D$ that differ only in the ranking of one voter i and satisfy $f(\succ') \succ_i f(\succ)$ (where $\succ_i$ is voter i's ranking in $\succ$). Note that both profiles must lie in the domain.

Fix a linear order $\lhd$ on A, called the *axis*. A ranking $\succ$ with top choice t is *single-peaked* with respect to $\lhd$ if $x \lhd y \trianglelefteq t$ or $t \trianglelefteq y \lhd x$ implies $y \succ x$. A profile is single-peaked with respect to $\lhd$ if all of its rankings are, and it is *single-peaked* if it is single-peaked with respect to some axis. A profile $(\succ_1, \dots, \succ_n)$ is *single-crossing* (with respect to this voter order) if for every pair of alternatives x, y the set $\{i : x \succ_i y\}$ is a prefix or a suffix of $(1, \dots, n)$.

Of the following, do {1., 8., 9.}, one from {2, 3}, and one from {4., 5., 6., 7.}

1. **Warm-ups.** Do 5 of the following. (Some of them refer to definitions that appear later in the assignment; feel free to skip ahead.)

   (a) *Transitivity of the majority relation (Black 1948).* Show that if $\succ$ is single-peaked and n is odd, then $\succ_{\text{maj}}$ is transitive. (Suppose $x \succ_{\text{maj}} y \succ_{\text{maj}} z \succ_{\text{maj}} x$ is a cycle, and let y' be the one of x, y, z in the middle of the axis. Two of the three majorities must intersect, since any two strict majorities share a voter. Find a voter who ranks y' last among the three, and explain why that contradicts single-peakedness.)

   (b) *The majority relation is single-peaked.* Let $\succ$ be single-peaked with respect to $\lhd$ with n odd. By (a), $\succ_{\text{maj}}$ is a strict linear order, so it is a ranking. Show that this ranking is single-peaked with respect to $\lhd$. (Let c be the Condorcet winner (the top of $\succ_{\text{maj}}$). Take $c \lhd b \lhd a$ and show $b \succ_{\text{maj}} a$. At least $(n+1)/2$ voters have their peak at or to the left of c. Why? The case $a \lhd b \lhd c$ is symmetric.)

   (c) *Single-crossing but not single-peaked.* Consider the profile with three voters

   $$a \succ_1 b \succ_1 c, \qquad c \succ_2 a \succ_2 b, \qquad c \succ_3 b \succ_3 a.$$

   i. Show that this profile is single-crossing.

   ii. Show that it is not single-peaked. (Who is ranked last by each voter?)

(d) *Single-peaked but not single-crossing.* Consider the profile with four voters

$$abcd, \qquad abdc, \qquad bacd, \qquad badc$$

(each written top to bottom).

i. Show that it is single-peaked with respect to $c \lhd b \lhd a \lhd d$.

ii. Show that it is not single-crossing with respect to *any* ordering of the four voters. (The set $\{i : a \succ_i b\} = \{1, 2\}$ must be the first two or the last two voters of the ordering. The set $\{i : c \succ_i d\} = \{1, 3\}$ must also be the first two or the last two. Is that possible?)

(e) *Single-crossing preferences have a representative voter.* Let $\succ$ be single-crossing with n odd, and let $h = (n+1)/2$. Show that $\succ_{\text{maj}} = \succ_h$, so the majority relation is the ranking of the median voter. In particular it is transitive. (For a pair x, y suppose voter h ranks x above y. The voters who rank x above y form a prefix or a suffix. How many voters does a prefix or a suffix that contains h have?)

(f) *Three definitions of single-peaked.* Let $\succ$ be a ranking with top choice t and let $\lhd$ be an axis. Prove that the following are equivalent:

(1) $\succ$ is single-peaked with respect to $\lhd$ (as defined above).

(2) (*no valley*) there are no $x \lhd y \lhd z$ with $x \succ y$ and $z \succ y$.

(3) for every k, the set of the k top-ranked alternatives of $\succ$ is an interval of $\lhd$.

(Prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). For the first, consider $y \trianglelefteq t$ and $t \trianglelefteq y$ separately.)

(g) *Counting.* Show that there are exactly 2^{m-1} rankings of m alternatives that are single-peaked with respect to a fixed axis. (Delete the last-ranked alternative. Where can it be on the axis?)

(h) *An even number of voters.*

i. Let $\succ$ be single-peaked and let n be even. Say that c is a *weak Condorcet winner* if no alternative o has $o \succ_{\text{maj}} c$. Show that a weak Condorcet winner exists. (Order the voters so that $\text{top}(\succ_1) \trianglelefteq \cdots \trianglelefteq \text{top}(\succ_n)$ and take $c = \text{top}(\succ_{n/2+1})$.)

ii. Say that $x \succeq_{\text{maj}} y$ if at least $n/2$ voters rank x above y. Show that the profile with four voters *bac*, *bac*, *cba*, *cba* is single-peaked with respect to $a \lhd b \lhd c$, but $\succeq_{\text{maj}}$ is not transitive. (Find x, y, z with $x \succeq_{\text{maj}} y \succeq_{\text{maj}} z$ but not $x \succeq_{\text{maj}} z$.)

(i) *A Condorcet winner has bounded distortion.* (This previews Problem 9, whose definitions you will need.) Suppose w is a Condorcet winner of $\succ$. Show that for every metric d on $N \cup A$ consistent with $\succ$, and for every $o \in A$, $\text{cost}(w) \leq 3\,\text{cost}(o)$. (Let N_w be the voters who rank w above o and N_o the rest, so $|N_o| < |N_w|$. Choose a permutation σ of the voters that maps N_o into N_w. For $v \in N_o$ and $u = \sigma(v)$, bound $d(v, w) \leq d(v, o) + d(o, u) + d(u, w)$ and use that u ranks w above o. Prove $d(v, w) \leq d(v, o) + 2d(\sigma(v), o)$ for every voter v, and then sum.)

(j) *No extension of f_{CW} to all profiles.* Let D_{CW} and f_{CW} be as in Problem 2, and let $|A| \geq 3$ and $n \geq 3$. Explain why f_{CW} cannot be extended to a voting rule on *all* profiles that is onto and IC. (Gibbard (1973) and Satterthwaite (1975). What would the extension have to be, and is f_{CW} dictatorial?)

(k) *Trees: a Condorcet winner, but a cyclic majority relation.* (Single-peakedness on a tree is defined in Problem 3.) Let T be a star with center x and leaves a, b, c (so the only edges of

the graph are $\{x, a\}$, $\{x, b\}$, and $\{x, c\}$). Consider the profile with the *three* voters $xabc$, $xbca$, $xcab$. Show that all three rankings are single-peaked on T, that x is the Condorcet winner, and that $a \succ_{\text{maj}} b \succ_{\text{maj}} c \succ_{\text{maj}} a$. Compare with warm-up 1a: the number of voters is odd, but the majority relation is not transitive.

(l) *Median rules with phantom voters (the easy direction of Moulin's theorem).* Let $A = \{1, \dots, m\}$ with the axis $1 \lhd 2 \lhd \cdots \lhd m$ (so we can compare alternatives with $<$ and $\leq$), and let D be the set of n-voter profiles that are single-peaked with respect to this axis. Write $t_i = \text{top}(\succ_i)$. For $2n-1$ numbers, med denotes the n-th smallest of them. A voting rule $f : D \to A$ is *tops-only* if $f(\succ) = f(\succ')$ whenever $\text{top}(\succ_i) = \text{top}(\succ'_i)$ for all i, and *Pareto* if $f(\succ) \neq b$ whenever every voter ranks some alternative a above b. Fix alternatives $\alpha_1, \dots, \alpha_{n-1} \in A$ (the *phantom voters*) and let

$$g(\succ) = \text{med}(t_1, \dots, t_n, \alpha_1, \dots, \alpha_{n-1}).$$

Show that g is tops-only, anonymous (as defined in Problem 4), Pareto, and IC.

(i) Check that g is tops-only and anonymous.

(ii) *Pareto.* Show that $\min_i t_i \leq g(\succ) \leq \max_i t_i$. (If $g(\succ) < \min_i t_i$, how many of the $2n-1$ points can be at most $g(\succ)$? How many are needed for $g(\succ)$ to be the n-th smallest?) Then show that if $y \in [\min_i t_i, \max_i t_i]$ and $z \neq y$, then some voter ranks y above z. (Use a voter with peak at most y and a voter with peak at least y.) Conclude that g is Pareto.

(iii) *IC.* Let $\succ, \succ' \in D$ differ only in voter i, and let $\mu = g(\succ)$ and $\mu' = g(\succ')$. Suppose $\mu < t_i$. Show that $\mu' \leq \mu$, and conclude that i does not prefer μ' to μ. Do the case $\mu > t_i$ by symmetry, and explain why the case $\mu = t_i$ is trivial. (Count the points that are at most μ and the points that are at least μ before and after the change.)

Problem 5 shows that these are the *only* such rules.

(m) *Checking an axis.* Let $\succ$ be a profile and let $\lhd$ be a linear order on A, given as a list.

(i) Use warm-up 1f(3) to show that a ranking $\succ_i$ is single-peaked with respect to $\lhd$ if and only if, when we read $\succ_i$ from top to bottom, each alternative after the first lies immediately to the left or immediately to the right of the interval formed by the alternatives read so far.

(ii) Give an algorithm that decides whether $\succ$ is single-peaked with respect to $\lhd$ in $O(nm)$ time. (Store the position of each alternative on the axis. For each voter keep the leftmost and rightmost position seen so far.)

(iii) For the three voters $e \succ d \succ b \succ c \succ a$, $e \succ d \succ c \succ b \succ a$, $d \succ e \succ c \succ a \succ b$, decide which of $a \lhd c \lhd e \lhd d \lhd b$ and $a \lhd b \lhd c \lhd d \lhd e$ is an axis. For the other one, name a voter and the step at which the check fails.

(n) *Borda has unbounded distortion.* Use the definitions of values, consistency and social welfare from Problem 8. The distortion of a deterministic rule f is $\sup_{\succ} \sup_v \max_a \text{sw}(a)/\text{sw}(f(\succ))$, where the ratio is infinite if $\text{sw}(f(\succ)) = 0$. The *Borda rule* gives a voter's k-th ranked alternative $m-k$ points and outputs an alternative with the most points in total.

(i) Let $A = \{w, x, y, z\}$ and consider the three voters $x \succ_1 w \succ_1 y \succ_1 z$, $y \succ_2 w \succ_2 z \succ_2 x$, $z \succ_3 w \succ_3 x \succ_3 y$. Compute the Borda scores and find the winner.

(ii) Let every voter give value 1 to her top choice and 0 to every other alternative. Check that this is consistent with the rankings, compute sw of every alternative, and conclude that Borda has unbounded distortion (Procaccia and Rosenschein 2006).

(iii) Contrast this with Plurality (choose an alternative that is ranked first by the most voters). Show that its winner w has $\mathrm{sw}(w) \geq n/m^2$, and conclude that its distortion is at most m^2 (Caragiannis and Procaccia 2011). (At least n/m voters rank w first. If w is a voter's top choice, why is $v_i(w) \geq 1/m$? Also $\mathrm{sw}(a) \leq n$ for all a.)

(o) *Plurality Veto by hand.* (Plurality Veto is defined in Problem 9(c).)

(i) Run Plurality Veto with the order $1, 2, 3$ on the three voters *bca*, *abc*, *cab* (the cycle from class). Write down the bijection σ and check that it is a perfect matching in the domination graph of the winner. Find another voter order that gives a different winner.

(ii) Plurality Veto is not Condorcet consistent. Show this using the three voters *abc*, *bac*, *cab*, and the order $1, 2, 3$. Does this contradict Problem 9(c) and warm-up 1i?

2. **The domain of Condorcet winners resists group manipulation.** Let D_{CW} be the set of profiles (for fixed n and A) that have a Condorcet winner, and let $f_{\mathrm{CW}} : D_{\mathrm{CW}} \to A$ return it. A *group manipulation* of a voting rule $f : D \to A$ consists of a nonempty set $S \subseteq N$ of voters and two profiles $\succ, \succ' \in D$ that differ only in the rankings of the voters in S, such that $f(\succ') \succ_i f(\succ)$ for every $i \in S$ (where $\succ_i$ is the ranking of i in $\succ$). We say f is *group strategyproof* if it has no group manipulation.

(a) Prove that f_{CW} is group strategyproof. In particular (take $|S| = 1$) it is IC. (Suppose $a = f_{\mathrm{CW}}(\succ)$ and $b = f_{\mathrm{CW}}(\succ')$ with $b \neq a$, and that everyone in S prefers b to a in $\succ$. Let M be the set of voters who rank a above b in $\succ$. Show $|M| > n/2$ and $M \cap S = \emptyset$. Then compare with the voters in M in $\succ'$.)

(b) Suppose n is odd. Explain why the median rule on single-peaked profiles (from class) is a special case of f_{CW}, and conclude that it is group strategyproof. (Note that group strategyproofness restricts to any subdomain of D_{CW}, for instance the profiles that are single-peaked with respect to a fixed axis.) Note that group strategyproofness is a stronger claim than what we proved in class.

3. **Single-peaked on a tree.** Let T be a tree with vertex set A. A ranking $\succ_i$ is *single-peaked on* T if for all $a \neq b$ such that a lies on the path in T from $\mathrm{top}(\succ_i)$ to b, we have $a \succ_i b$. A profile is single-peaked on T if all of its rankings are.

(a) Show that if T is a path then this agrees with single-peakedness with respect to the axis given by the path.

(b) *No valleys.* Show that if b lies on the path between a and c in T, then no ranking that is single-peaked on T has both $a \succ_i b$ and $c \succ_i b$. (Let t be the top choice and let p be the vertex of the a–c path that is closest to t. The paths from t to a and from t to c both pass through p. Where is b relative to p?)

(c) *Theorem (Demange 1982).* Show that if $\succ$ is single-peaked on T and n is odd, then $\succ$ has a Condorcet winner.

i. For every edge $\{a, b\}$ of T exactly one of $a \succ_{\mathrm{maj}} b$ and $b \succ_{\mathrm{maj}} a$ holds. Orient each edge accordingly. Show that the resulting directed graph has a vertex c with no incoming edges. (An orientation of a tree has no directed cycle.)

ii. Show that c is a Condorcet winner. (Let $d \neq c$ and let $c, x, \ldots, d$ be the path from c to d. A strict majority prefers c to x. Use (b) to show that each of these voters prefers x to d.)

4. **Only the Condorcet rule is anonymous, neutral, Pareto, and IC on D_{CW}.** Throughout this problem n is odd and $D = D_{\text{CW}}$. A voting rule $f : D \to A$ is

   - *anonymous* if $f(\succ) = f(\succ \circ \pi)$ for every permutation π of the voters (so the output only depends on the multiset of rankings);
   - *neutral* if $f(\sigma(\succ)) = \sigma(f(\succ))$ for every permutation σ of the alternatives, where $\sigma(\succ)$ renames the alternatives in every ranking of $\succ$;
   - *Pareto* if whenever every voter ranks a above b, we have $f(\succ) \neq b$.

   We will prove: *if f is anonymous, neutral, Pareto, and IC then $f = f_{\text{CW}}$.*

   (a) *f_{CW} has these properties.* Check that f_{CW} is anonymous, neutral, and Pareto. (It is IC by Problem 2.)

   (b) *A monotonicity lemma.* Let f be IC, and let $\succ, \succ' \in D$ differ only in the ranking of voter i ($\succ_i$ in $\succ$ and $\succ'_i$ in $\succ'$). Suppose $f(\succ) = x$ and that x only moved up for voter i, meaning $\{y : y \succ'_i x\} \subseteq \{y : y \succ_i x\}$. Show that $f(\succ') = x$. (Suppose $f(\succ') = y \neq x$. Apply IC once with $\succ_i$ as the truth and once with $\succ'_i$ as the truth.)

   (c) *Two kinds of voters.* Fix two alternatives $c \neq x$ and a ranking r of the other alternatives. A voter of *type C* has the ranking $c \succ x \succ r$ and a voter of *type X* has the ranking $x \succ c \succ r$. For $k = 0, \ldots, n$ let $\succ^{(k)}$ be a profile with k voters of type C and $n-k$ of type X. Show that $\succ^{(k)} \in D$ (who is the Condorcet winner?), and that by anonymity $f(\succ^{(k)})$ does not depend on which voters are of which type.

   (d) *The outcome jumps at most once, and only from x to c.* Show that if $f(\succ^{(k)}) \neq f(Q_{k+1})$ then $f(\succ^{(k)}) = x$ and $f(Q_{k+1}) = c$. (The two profiles differ in one voter, who switches from type X to type C. Apply IC in both directions, and note that the two rankings only differ on the pair $\{x, c\}$.) Use Pareto to show $f(Q_0) = x$ and $f(Q_n) = c$. Conclude that there is a $k^* \in \{0, \ldots, n-1\}$ with $f(\succ^{(k)}) = x$ for $k \leq k^*$ and $f(\succ^{(k)}) = c$ for $k > k^*$.

   (e) *Neutrality pins down the threshold.* Let σ swap c and x and fix all other alternatives. Show $\sigma(\succ^{(k)}) = \succ^{(n-k)}$, and use neutrality to prove $k^* = (n-1)/2$. Conclude that $f(\succ^{(k)}) = c$ exactly when $k > n/2$, that is, exactly when c is the Condorcet winner of $\succ^{(k)}$.

   (f) *Every profile.* Let $\succ \in D$ have Condorcet winner c, and suppose $f(\succ) = x \neq c$. Let M be the set of voters who rank c above x in $\succ$ (so $|M| > n/2$). Let $\succ^*$ be the profile in which each voter in M is of type C and each voter not in M is of type X (for a fixed r).

      i. Change the voters from their ranking in $\succ$ to their ranking in $\succ^*$ one at a time. Show that c is still a Condorcet winner after each step, so each intermediate profile is in D. (For which alternatives y can the change of one voter alter whether the voter ranks c above y, and in which direction?)

      ii. Show that in each step, x only moves up for the voter being changed. Use (b) to conclude $f(\succ^*) = x$.

      iii. Derive a contradiction with (e), and conclude $f = f_{\text{CW}}$.

   (g) *Hypotheses.* For each of anonymity, neutrality, Pareto, and "n odd", say where it was used. Then show that neutrality cannot simply be dropped: for $A = \{a, b\}$ and $n = 5$ (here D is the set of all profiles), give a voting rule that is anonymous, Pareto, and IC but is not f_{CW}. (Let a win exactly when at least two voters rank it first.)

Remark (not to be proved). The same conclusion holds under much weaker assumptions: for n odd and $|A| \geq 3$, any IC voting rule on D_{CW} that is onto and non-dictatorial equals f_{CW} (Campbell and Kelly 2002, 2016). For an even number of voters, the analogous result holds with anonymity and neutrality (Campbell and Kelly 2015) or with anonymity and Pareto (Peters 2019).

5. **Moulin's characterization (tops-only version).** Use the notation of warm-up 1l: $A = \{1, \ldots, m\}$ with the axis $1 \lhd 2 \lhd \cdots \lhd m$, D is the set of n-voter profiles that are single-peaked with respect to this axis, $t_i = \text{top}(\succ_i)$, and med is the n-th smallest of $2n-1$ numbers.

   Theorem (Moulin 1980). A tops-only rule $f : D \to A$ is anonymous, Pareto, and IC if and only if there are alternatives $\alpha_1, \ldots, \alpha_{n-1} \in A$ such that for every $\succ \in D$,

   $$f(\succ) = \text{med}(t_1, \ldots, t_n, \alpha_1, \ldots, \alpha_{n-1}).$$

   The hypothesis "tops-only" can be removed (Barberà and Jackson 1994), but we will not do this. The "if" direction is warm-up 1l. We prove the "only if" direction: let f be tops-only, anonymous, Pareto, and IC.

   (a) *An observation about single-peaked rankings.* Let t be an alternative and let $y \neq z$ be alternatives with $y \neq t$. Suppose that $z = t$, or z lies on the other side of t than y, or z lies on the same side of t as y but closer to t. Show that some ranking that is single-peaked with respect to $\lhd$ and has top choice t ranks z above y. (Single-peaked rankings can be built by starting at t and repeatedly adding the alternative just to the left or just to the right of the current interval. Choose which side to extend first.)

   (b) *Misreports move the outcome away from the peak.* Let $\succ, \succ' \in D$ differ only in voter i, and let $y = f(\succ)$ and $z = f(\succ')$. Show that if $y > t_i$ then $z \geq y$, and if $y < t_i$ then $z \leq y$. (Use (a) and IC. Since f is tops-only, you may take voter i's true ranking to be any single-peaked ranking with peak t_i.)

   (c) *Outcomes lie between the peaks.* Show that $\min_i t_i \leq f(\succ) \leq \max_i t_i$. (If $f(\succ) < \min_i t_i$, how does every voter rank $\min_i t_i$ versus $f(\succ)$?)

   (d) *The key claim.* Fix an alternative $x > 1$. Let $\succ, \succ' \in D$ differ only in voter i, and assume that $t_i \geq x$ implies $t'_i \geq x$. Show that if $f(\succ) \geq x$ then $f(\succ') \geq x$. (Suppose $f(\succ) = y \geq x$ and $f(\succ') = y' < x$. If $t'_i \geq x$, apply (b) with $\succ'$ as the truth. Otherwise $t'_i < x$ and so $t_i < x$, and you can apply (b) with $\succ$ as the truth.)

   (e) *Thresholds.* For $x > 1$ let $s(\succ) = |\{i : t_i \geq x\}|$. Show that there is a number $q_x \in \{1, \ldots, n\}$ such that $f(\succ) \geq x$ if and only if $s(\succ) \geq q_x$. (Use anonymity to permute the voters of a profile so that $\{i : t_i \geq x\}$ becomes whichever set of size s you like. Then use (d) to change voters one at a time. Use (c) for $1 \leq q_x \leq n$.)

   (f) *Reading off the phantoms.* For $k = 1, \ldots, n-1$ let $e_k \in D$ be a profile in which k voters have peak m and $n-k$ voters have peak 1, and let $\alpha_k = f(e_k)$. Show that for every $x > 1$ the number of $k \in \{1, \ldots, n-1\}$ with $\alpha_k \geq x$ is exactly $n - q_x$.

   (g) *Conclusion.* Fix $\succ \in D$ and $x > 1$. Show that $f(\succ) \geq x$ if and only if at least n of the $2n-1$ numbers $t_1, \ldots, t_n, \alpha_1, \ldots, \alpha_{n-1}$ are at least x, and conclude that $f(\succ) = \text{med}(t_1, \ldots, t_n, \alpha_1, \ldots, \alpha_{n-1})$.

6. **Single-crossing profiles.** Throughout, n is odd, $\succ = (\succ_1, \ldots, \succ_n)$ is single-crossing with respect to the order $1, \ldots, n$, and $h = (n+1)/2$.

(a) Show that $\mathrm{top}(\succ_h)$ is the Condorcet winner of $\succ$.

(b) *Incentive compatibility, two ways.* Let $f_{\mathrm{med}}(\succ) = \mathrm{top}(\succ_h)$.

i. Suppose the order of the voters is publicly known and fixed, and a report is allowed if the resulting profile is single-crossing *with respect to this fixed order*. Show that f_{med} is group strategyproof. (Who can change the outcome?)

ii. Now suppose a report is allowed if the resulting profile is single-crossing with respect to *some* order of the voters, so a voter may move to a different position. Let $f_{\mathrm{SC}}(\succ)$ be the top choice of the median voter in an order in which $\succ$ is single-crossing. Show that f_{SC} is well defined (the answer does not depend on which valid order is used), and that it is group strategyproof. (Use (a) and Problem 2.)

(c) *Finding the order.* Let $d(\succ, \succ')$ be the number of pairs $\{x, y\}$ of alternatives that the two rankings order differently. For each j let D_j be the set of pairs that $\succ_j$ and $\succ_1$ order differently.

i. Show that $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_n$.

ii. Show that for $i < j$, $d(\succ_i, \succ_j) = |D_j| - |D_i|$, and that if $\succ_i \neq \succ_j$ then $|D_i| < |D_j|$.

iii. Show that if (u, w) is a pair of voters at maximum distance, then $\{\succ_u, \succ_w\} = \{\succ_1, \succ_n\}$.

iv. Give an algorithm that, given an arbitrary profile, either outputs an order of the voters in which it is single-crossing or correctly reports that none exists. (Take a maximum-distance pair (u, w), sort the voters by $d(\succ_u, \cdot)$, and then check the result directly.) Explain why it is correct (if the profile is single-crossing in some order, why does the algorithm find one?) and bound its running time.

v. Show that if all the rankings in $\succ$ are distinct, the order making $\succ$ single-crossing is unique up to reversal.

Remark (not to be proved). Moulin's characterization in Problem 5 has an analogue on single-crossing domains: Saporiti (2009) shows that anonymous, unanimous, IC rules are again generalized median rules with phantom voters (see Section 3.3 of the survey for the precise statement).

7. **Finding an axis.** Given a profile $\succ$, we want to decide whether $\succ$ is single-peaked and, if so, find an axis. By warm-up 1f, $\succ$ is single-peaked with respect to $\lhd$ if and only if for every voter i and every k, the k top-ranked alternatives of $\succ_i$ form an interval of $\lhd$.

(a) *Reduction to consecutive ones.* A 0–1 matrix has the *consecutive ones property* if its columns can be permuted so that in every row the ones are consecutive. Build the matrix whose columns are the alternatives and which has, for every voter i and every $k = 1, \ldots, m$, the row indicating the k top-ranked alternatives of $\succ_i$. Show that $\succ$ is single-peaked if and only if this matrix has the consecutive ones property, and that every valid column permutation is an axis. Write down the matrix for $v_1 : \; b \succ c \succ a \succ d$ and $v_2 : \; c \succ d \succ a \succ b$ and show that $\{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}$ cannot all be intervals of one axis; so this profile is not single-peaked. Booth and Lueker (1976) give a linear-time algorithm for the consecutive ones problem; what does that give for single-peakedness?

(b) *Bottom alternatives.* Show that if $\succ$ is single-peaked with respect to $\lhd$, then every voter's last-ranked alternative is the leftmost or the rightmost alternative of $\lhd$. Conclude that if three or more different alternatives are ranked last by some voter, then $\succ$ is not single-peaked, and that if exactly two are, they must be the two ends of any axis. Show that

deleting a leftmost or rightmost alternative of the axis from every ranking leaves a profile that is single-peaked on the remaining axis.

This suggests building the axis from the outside in. Keep a list *Left* and a list *Right* (initially empty) and the set A' of unplaced alternatives (initially A). When we say "bottom alternatives" we mean those ranked last among A' by some voter.

- *Stage 1* (while *Right* is empty): let B be the set of bottom alternatives. If $|B| \geq 3$, reject. If $B = \{x\}$, append x to the end of *Left*. If $B = \{x, y\}$, append x to the end of *Left* and put y at the beginning of *Right*. Remove B from A'.
- *Stage 2* (while $|A'| \geq 2$): let ℓ be the last element of *Left* and r the first element of *Right*, and let B be the set of bottom alternatives (reject if $|B| \geq 3$). A voter is *x-dissenting* if they do not rank x first among A'. Let L be the set of $x \in B$ for which some x-dissenting voter has $r \succ_i x \succ_i \ell$, and let R be the set of $x \in B$ for which some x-dissenting voter has $\ell \succ_i x \succ_i r$. Reject if $|L| > 1$, $|R| > 1$, or $L \cap R \neq \emptyset$. Assign the elements of $B \setminus (L \cup R)$ arbitrarily to L or R so that $|L| \leq 1$ and $|R| \leq 1$. Append L to *Left*, put R at the beginning of *Right*, and remove B from A'.
- Output *Left*, then the remaining element of A' (if any), then *Right*.

(c) *The forcing rule.* In Stage 2, suppose $x \in B$ and some x-dissenting voter i has $r \succ_i x \succ_i \ell$. Show that in every axis consistent with the partial axis so far, x must be placed immediately after ℓ. (Suppose instead x is placed later. Look at i's favorite alternative t in A' (which is not x) and the three alternatives ℓ, x, t. Where must the unplaced alternatives lie on the axis?) State the symmetric rule. Explain why the algorithm rejects when $|L| > 1$ or $L \cap R \neq \emptyset$ (what do the rules force?).

(d) *Run it.* Run the algorithm on the three voters $e \succ d \succ b \succ c \succ a$, $e \succ d \succ c \succ b \succ a$, $d \succ e \succ c \succ a \succ b$. Which choices were forced and which were arbitrary? Write down every axis this profile is single-peaked with respect to.

(e) *Running time.* Show that a direct implementation takes $O(nm^2)$ time. (There are at most m rounds. How long does one round take?) (Bonus: with pointers to each voter's current top and bottom alternative in A' and a table of positions, each round takes $O(n)$ time, for $O(nm)$ in total.)

(f) *Many axes.* Call $A'' \subseteq A$ a *common prefix* if every voter ranks all alternatives in A'' above all alternatives outside A''. Show that if $\succ$ is single-peaked with respect to $\lhd$ and A'' is a common prefix, then A'' is an interval of $\lhd$, and reversing $\lhd$ on that interval gives another axis for $\succ$. Explain how this accounts for the arbitrary choices in (e).

Remark (not required). The proof that the algorithm never outputs a wrong axis when it does not reject is more involved; see Section 4.1 of the survey (Doignon and Falmagne 1994, Escoffier, Lang and Öztürk 2008).

Why you can trust the output anyway. If the algorithm outputs an axis, then by warm-up 1m you can verify in $O(nm)$ time that the profile really is single-peaked with respect to it. So the only part of the correctness proof that one has to take on faith is that the algorithm does not reject a profile that is single-peaked (and the rejection rules in (b) and (c) are the ones that prove this).

8. **Randomized distortion.** In this problem voters have *values*: voter i has $v_i : A \to [0, 1]$ with $\sum_{a \in A} v_i(a) = 1$, and v_i is *consistent* with $\succ_i$ if $v_i(a) > v_i(b)$ implies $a \succ_i b$. The *social*

welfare of a is $\mathrm{sw}(a) = \sum_i v_i(a)$. A *randomized* voting rule maps a profile $\succ$ to a probability distribution over A. Its distortion is

$$\Delta(f) = \sup_{\succ} \sup_{v \text{ consistent with } \succ} \frac{\max_a \mathrm{sw}(a)}{\mathbb{E}[\mathrm{sw}(f(\succ))]}.$$

Recall that in class we showed that $\Delta(f) \geq \sqrt{m}/3$ for every randomized rule f. Now let's come up with a reasonably close upper bound.

The *harmonic score* of a is $\mathrm{sc}(a) = \sum_i 1/\mathrm{rank}_i(a)$, where $\mathrm{rank}_i(a)$ is the position of a in $\succ_i$ (so the top choice has rank 1). Define the randomized rule f that, with probability $1/2$, outputs a uniformly random alternative, and with probability $1/2$ outputs a with probability $\mathrm{sc}(a)/\sum_b \mathrm{sc}(b)$.

(a) Show that $\mathrm{sw}(a) \leq \mathrm{sc}(a)$ for every a and every consistent v. (If a has rank k in $\succ_i$, how many alternatives have value at least $v_i(a)$? How large can $k \cdot v_i(a)$ be?)

(b) Show that $\sum_a \mathrm{sc}(a) \leq n(\ln m + 1)$. ($\sum_{k=1}^{m} 1/k \leq 1 + \ln m$.)

(c) Show that $\mathbb{E}[\mathrm{sw}(f(\succ))] \geq n/(2m)$. (What is the expected social welfare of a uniformly random alternative? Use $\sum_a \mathrm{sw}(a) = n$.)

(d) Let $\theta = n\sqrt{(\ln m + 1)/m}$ and fix a. Show that if $\mathrm{sc}(a) < \theta$ then $\mathrm{sw}(a)/\mathbb{E}[\mathrm{sw}(f(\succ))] \leq 2\sqrt{m(\ln m + 1)}$.

(e) Show that if $\mathrm{sc}(a) \geq \theta$ then $\Pr[f(\succ) = a] \geq 1/(2\sqrt{m(\ln m + 1)})$, and deduce that $\mathrm{sw}(a)/\mathbb{E}[\mathrm{sw}(f(\succ))] \leq 2\sqrt{m(\ln m + 1)}$ in this case as well. ($\mathbb{E}[\mathrm{sw}(f(\succ))] \geq \Pr[f(\succ) = a] \cdot \mathrm{sw}(a)$.)

(f) Conclude that $\Delta(f) \leq 2\sqrt{m(\ln m + 1)}$. Compare with the lower bound from class.

Remark (not required). Boutilier, Caragiannis, Haber, Lu, Procaccia, and Sheffet (2015) show that the gap can be closed to a factor of $\log^* m$ with a more careful analysis.

9. **Metric distortion.** Suppose the voters and the alternatives (here also called candidates) are points of a metric space, that is, there is a distance d on $N \cup A$ that is symmetric and satisfies the triangle inequality. The voting rule only sees the rankings, but the voters' rankings come from the distances: d is *consistent* with $\succ$ if $c \succ_i c'$ implies $d(i,c) \leq d(i,c')$ for all voters i (ties in distance are allowed). The *cost* of a candidate is $\mathrm{cost}(c) = \sum_{i \in N} d(i,c)$. The *distortion* of a rule f is

$$\sup_{\succ} \sup_{d \text{ consistent with } \succ} \frac{\mathrm{cost}(f(\succ))}{\min_{c \in A} \mathrm{cost}(c)}.$$

(All of this is about deterministic rules, with strict rankings, and the metric is chosen after the rule.)

(a) *Lower bound.* Show that every deterministic voting rule has distortion at least 3. Assume n is even.

   i. Consider a profile in which half of the voters rank $a \succ b$ (and everything else arbitrarily) and half rank $b \succ a$. WLOG the rule outputs a on this profile. (If it outputs b, swap the roles. If $m > 2$, rank the other alternatives last and place them very far from everyone, so that outputting one of them is even worse.)

   ii. Put a and b at distance 2 on a line, put the voters who rank b first at b, and the other voters at the midpoint. Check that this metric is consistent with the profile, and compute $\mathrm{cost}(a)$ and $\mathrm{cost}(b)$.

iii. Conclude the distortion is at least 3. Then modify the metric slightly to make it strictly consistent (no ties), and show that the ratio approaches 3.

(b) *The domination lemma.* For a candidate w, the *domination graph* $G(w)$ is the bipartite graph with two copies of N as its sides, and an edge from voter i on the left to voter j on the right whenever i ranks w at least as high as $\mathrm{top}(\succ_j)$. Show that if $G(w)$ has a perfect matching σ, then $\mathrm{cost}(w) \le 3\,\mathrm{cost}(o)$ for every candidate o and every consistent metric. (For a voter i, let $j = \sigma(i)$ and $t = \mathrm{top}(\succ_j)$. Show $d(i,w) \le d(i,t)$, then bound $d(i,t) \le d(i,o) + d(o,j) + d(j,t)$ and use $d(j,t) \le d(j,o)$. Conclude $d(i,w) \le d(i,o) + 2d(j,o)$, and sum over i.)

(c) *Plurality Veto* (Kizilkaya and Kempe 2022). Set $\mathrm{score}(c)$ to be the number of voters whose top choice is c. Fix any order $v_1, \dots, v_n$ of the voters. For $k = 1, \dots, n$: let A_k be the set of candidates with positive score, let c_k be the candidate in A_k that v_k ranks lowest, and decrease $\mathrm{score}(c_k)$ by one. Output c_n.

i. Show that all scores stay nonnegative and that all scores are 0 at the end. (What is the sum of the scores at the start, and how does it change?)

ii. Show that each candidate c is chosen as c_k for exactly as many k as there are voters with top choice c.

iii. Show that $c_n \in A_k$ for every k.

iv. Use (b) to define a bijection σ on voters with $\mathrm{top}(\succ_{\sigma(v_k)}) = c_k$ for all k. Show that $\{(v_k, \sigma(v_k))\}$ is a perfect matching in $G(c_n)$.

v. Conclude that Plurality Veto has distortion at most 3, whichever order is used. Together with part (a), this shows that 3 is the best possible distortion for deterministic rules.

(d) *Randomization.* A *randomized* rule maps each profile to a probability distribution over A, and its distortion is $\sup_{\succ} \sup_d \mathbb{E}[\mathrm{cost}(f(\succ))] / \min_{c \in A} \mathrm{cost}(c)$, where d ranges over metrics consistent with $\succ$.

(i) *Lower bound.* Show that every randomized rule has distortion at least 2. (Use the profile from (a), and let p be the probability that the rule outputs a on it. Take the metric from (a)(ii) and also its mirror image (with the roles of a and b swapped). Compute the expected cost over the optimal cost in both and show that the larger of the two ratios is at least 2 for every p. For $m > 2$ rank the other alternatives last and put them very far away; you may also just take $m = 2$.)

(ii) *Random Dictatorship.* Pick a voter uniformly at random and output her top choice. Show that its distortion is at most $3 - 2/n$. (Fix $o \in A$, let $C = \mathrm{cost}(o)$, and let $t_j = \mathrm{top}(\succ_j)$. The expected cost is $\frac{1}{n}\sum_j \sum_i d(i, t_j)$. For $i \ne j$ use $d(i,t_j) \le d(i,o) + d(o,j) + d(j,t_j) \le d(i,o) + 2d(j,o)$. For $i = j$ use the better bound $d(i,t_i) \le d(i,o)$. Sum everything.)

(iii) Compare (i) and (ii) with the deterministic bound 3. How much does randomization help when n is large? (The best possible distortion of randomized rules in the metric setting was listed as an open problem in the survey of Anshelevich, Filos-Ratsikas, Shah, and Voudouris 2021; Random Dictatorship is analyzed in Anshelevich and Postl 2017.)

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