Social Choice Theory

Cleaned Mathematics Notes

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1 Social Welfare Functions and Domain Restrictions

Let $\mathcal{A}$ denote the set of alternatives, with $|\mathcal{A}| \geq 3$. A *preference profile* is $\succ = (\succ_1, \ldots, \succ_n)$, and a social welfare function F maps profiles into a social ranking.

Theorem 1 (Arrow's impossibility theorem)**.** *For $|\mathcal{A}| \geq 3$, if a social welfare function F is defined on the unrestricted domain of preference profiles and satisfies unanimity (Pareto) and independence of irrelevant alternatives (IIA), then F is dictatorial (assuming social rankings are complete and transitive).*

Theorem 2 (Gibbard–Satterthwaite theorem)**.** *For $|\mathcal{A}| \geq 3$, any onto, strategy-proof social choice function f on the unrestricted domain of strict rankings is dictatorial.*

Key distinction: Arrow's theorem concerns a *social welfare function* producing a ranking; Gibbard–Satterthwaite concerns a *social choice function* selecting a winner.

2 Condorcet Winners and the Condorcet Paradox

Definition 1 (Condorcet winner)**.** *An alternative $a \in \mathcal{A}$ is a* (weak) Condorcet winner *if, for every other alternative $b \neq a$, at least half of the voters weakly favor a over b in their pairwise comparison. A* strict Condorcet winner *defeats every other alternative by a strict majority.*

Condorcet cycle example:

Voter 1	Voter 2	Voter 3
a	c	b
b	a	c
c	b	a

Here a beats b by 2–1, b beats c by 2–1, and c beats a by 2–1. Hence majority preference cycles and there is no Condorcet winner.

2.1 A Condorcet-winner domain restriction

Let $\mathcal{D}_{\mathrm{CW}}$ be a domain containing only profiles with a Condorcet winner. Define $f(\succ)$ to select that winner (using a specified tie rule if needed).

Proposition 1. *On a domain of profiles with a* strict *Condorcet winner, the Condorcet-winner selection rule is incentive compatible against unilateral misreports that stay in the domain.*

Proof. Let $a = f(\succ)$ and suppose voter i changes only their ranking, giving $\succ'$. For any $b \neq a$, the original strict majority for a over b cannot be reversed by voter i if voter i genuinely prefers b to a: that voter already counted against a in the original pairwise contest. Therefore, if the new winner is b, voter i cannot truly prefer b to a. The deviation is not profitable. □

Note: The handwritten argument uses a weak Condorcet winner. With ties, the precise claim depends on how ties are resolved; the strict version above avoids this ambiguity.

3 Single-Peaked Preferences

Fix a linear ordering $\triangleleft$ of the alternatives (the *policy axis*).

Definition 2 (Single-peaked ranking)**.** *A ranking $\succ_i$ is single-peaked with respect to $\triangleleft$ if there is a top alternative (peak) t_i such that, moving away from t_i along either side of the axis, alternatives become less preferred. In particular, if $x \triangleleft y \triangleleft t_i$ or $t_i \triangleleft y \triangleleft x$, then $y \succ_i x$.*

Example. With policy axis $c \triangleleft d \triangleleft a \triangleleft b$, the following rankings are single-peaked:

Voter 1	Voter 2	Voter 3	Voter 4
a	b	b	b
b	a	a	a
d	d	d	d
c	c	c	c

Transcription note: The original page contains two small ranking tables. This table preserves the clearest readable example; the other handwritten example is less legible.

Proposition 2 (Median-voter result)**.** *Every profile of single-peaked rankings has a weak Condorcet winner.*

Proof. Let $t_1, \ldots, t_n$ be voters' peaks, ordered along the policy axis so that $t_{(1)} \trianglelefteq \cdots \trianglelefteq t_{(n)}$. Choose a median peak t_m, where $m = \lceil n/2 \rceil$. For any alternative a to the right of t_m, at least half the voters have peaks at or to the left of t_m, and thus prefer t_m to a. The symmetric argument applies to alternatives left of t_m. Therefore t_m is a weak Condorcet winner. □

3.1 Strategy-proofness of the median-peak rule

Define $f(\succ)$ to be the median of the reported peaks (with a fixed median convention for even n).

Proposition 3. *The median-peak rule is incentive compatible on the single-peaked domain.*

Proof. Fix voter i and the other voters' peaks. As voter i changes their reported peak, the selected median moves monotonically along the axis, but it cannot be moved closer to voter i's true peak than the outcome obtained by reporting that peak. Any deviation either leaves the outcome unchanged or moves it farther from the true peak on the same side. By single-peakedness, the deviation cannot improve voter i's outcome. □

Ranking table visible in the third photograph:

1	2	3
b	a	c
c	b	a
a	c	b