Distortion in Voting

CS 386: Algorithmic Game Theory I

November 5, 2026

Contents

1 The model and the definition of distortion 1
2 A lower bound for deterministic rules 2
3 Plurality's upper bound 3
4 A lower bound for randomized rules 3
5 Randomized upper bounds stated in lecture 4

1 The model and the definition of distortion

Let $A = \{a_1, \ldots, a_m\}$ be the set of candidates, and let there be n voters. Each voter i has a *unit-sum valuation* $v_i : A \to [0, 1]$, meaning that

$$\sum_{a \in A} v_i(a) = 1.$$

For a valuation profile $v = (v_1, \ldots, v_n)$, the social welfare of candidate a and the optimal welfare are

$$\mathrm{sw}_v(a) = \sum_{i=1}^{n} v_i(a), \qquad \mathrm{OPT}(v) = \max_{a \in A} \mathrm{sw}_v(a).$$

The objective is to choose a candidate with large social welfare using only the voters' rankings. Normalization gives $\sum_{a \in A} \mathrm{sw}_v(a) = n$, so $n/m \le \mathrm{OPT}(v) \le n$.

A ranking profile is $\prec = (\prec_1, \ldots, \prec_n)$, where $b \prec_i a$ means that voter i ranks a above b. A valuation is *compatible* with the ranking if

$$b \prec_i a \implies v_i(b) \le v_i(a).$$

Equal values may be ordered arbitrarily on a ballot. Write $V(\prec)$ for the set of unit-sum valuation profiles compatible with $\prec$.

An *ordinal social choice rule* f uses $\prec$ to select a winner. A randomized rule chooses a with probability $p_\prec(a) = \Pr[f(\prec) = a]$; a deterministic rule puts probability one on its winner.

Definition 1 (Distortion)**.** The distortion of f is

$$\Delta(f) = \sup_{\prec} \sup_{v \in V(\prec)} \frac{\mathrm{OPT}(v)}{\mathbb{E}[\mathrm{sw}_v(f(\prec))]}, \qquad \mathbb{E}[\mathrm{sw}_v(f(\prec))] = \sum_{a \in A} p_\prec(a)\, \mathrm{sw}_v(a).$$

A zero denominator gives infinite distortion. In the expectation, the valuations are fixed and only the rule's random draw varies. For a deterministic rule, the denominator is the welfare of its selected candidate.

2 A lower bound for deterministic rules

Theorem 1 (Deterministic distortion)**.** *For $m \geq 4$ and $n \geq m-2$, every deterministic ordinal rule has distortion $\Omega(m^2)$. If m is even and $m-2$ divides n, the equal-group construction gives $\Delta(f) \geq m(m-2)/4$.*

Proof. First take $g = m-2$ even and $q = n/g$ integral. Partition the voters into g groups $N_1, \ldots, N_g$, each of size q. Fix the following rankings, with the remaining positions filled in a fixed order.

Groups N_j	First choice	Second choice
First block: $1 \leq j \leq g/2$	a_j	a_{m-1}
Second block: $g/2 < j \leq g$	a_j	a_m

Each candidate appears in the top two positions in exactly one block. Let $w = f(\prec)$. Let $a^\star$ be the shared second choice in the block opposite w's block. Choose one group U in w's block: if $w = a_j$ for $j \leq g$, take $U = N_j$; if w is the shared second choice, take any group in that block.

Choose the following valuations after the ranking profile is fixed. Values omitted from the table are zero.

Voters	Values
In the block opposite w	$1/2$ for each of their first two choices
In U	$1/m$ for every candidate
All remaining voters	1 for their first choice

Each valuation sums to one and is compatible with the ballot. Only the uniform group U gives positive value to w: the opposite block has w outside its top two, and the other voters in w's block value only their distinct first choices. Thus

$$\mathrm{sw}_v(w) = \frac{q}{m}, \qquad \mathrm{sw}_v(a^\star) = \frac{q}{m} + \frac{g}{2} q \cdot \frac{1}{2} = \frac{q}{m} + \frac{n}{4}.$$

For the board's case $w = a_{m-2}$, this gives $U = N_{m-2}$ and $a^\star = a_{m-1}$; the candidate a_m also has welfare q/m. In every case,

$$\Delta(f) \geq \frac{\mathrm{sw}_v(a^\star)}{\mathrm{sw}_v(w)} \geq \frac{n/4}{q/m} = \frac{m(m-2)}{4}.$$

For general $m \geq 4$ and $n \geq g$, use g groups whose sizes differ by at most one, and split the groups into two blocks of sizes $\lfloor g/2 \rfloor$ and $\lceil g/2 \rceil$. Every group has between $n/(2g)$ and $2n/g$ voters. Each block has at least $g/3$ groups and hence at least $n/6$ voters. With the same valuations, $\mathrm{sw}_v(a^\star) \geq n/12$ and $\mathrm{sw}_v(w) \leq 2n/(gm)$. Their ratio is at least $gm/24 = \Omega(m^2)$. □

3 Plurality's upper bound

Proposition 2. *Plurality has distortion at most m^2. Its worst-case order in m matches the deterministic lower bound.*

Proof. The plurality winner w receives at least n/m first-place votes, since the n first-place votes are split among m candidates. A voter's first choice has value at least $1/m$: it is a maximum among m nonnegative values summing to one. Therefore,

$$\mathrm{sw}_v(w) \geq \frac{n}{m} \cdot \frac{1}{m} = \frac{n}{m^2}.$$

Since each voter gives any one candidate value at most one, $\mathrm{OPT}(v) \leq n$. Consequently

$$\frac{\mathrm{OPT}(v)}{\mathrm{sw}_v(w)} \leq \frac{n}{n/m^2} = m^2.$$

The preceding theorem supplies the matching lower order in its range of electorate sizes. □

4 A lower bound for randomized rules

Theorem 3 (Randomized distortion)**.** *For $m \geq 2$ and $n \geq \sqrt{m}$, every randomized ordinal rule has distortion $\Omega(\sqrt{m})$. When $m = t^2$ and t divides n, the equal-group construction gives $\Delta(f) \geq t/3$.*

Proof. Start with $t = \sqrt{m}$ an integer and let $q = n/t$. Partition the voters into groups $N_1, \ldots, N_t$ of size q. In group N_j, rank a_j first and fill the rest of the ballot in any fixed order. At this profile,

$$\sum_{j=1}^{t} \Pr[f(\prec) = a_j] \leq 1,$$

so some $a^\star = a_{j^\star}$ is selected with probability $p \leq 1/t$.

Give each voter in $N_{j^\star}$ value one for $a^\star$ and zero for every other candidate. Give every other voter value $1/m$ for every candidate. These valuations are compatible with the fixed rankings. The choice of valuations uses the selection probabilities, before any winner is drawn.

The resulting welfares are

$$\mathrm{sw}_v(a^\star) = q + \frac{n-q}{m}, \qquad \mathrm{sw}_v(a) = \frac{n-q}{m} \quad (a \neq a^\star).$$

In particular, $q \leq \mathrm{sw}_v(a^\star) \leq 2q$, because $n/m \leq n/t = q$. Every other candidate has welfare at most n/m. The expected welfare of the rule is therefore bounded by

$$\mathbb{E}[\mathrm{sw}_v(f(\prec))] \leq p(2q) + (1-p)\frac{n}{m} \leq \frac{2q}{t} + \frac{n}{m} = \frac{3n}{m}.$$

As $\mathrm{OPT}(v) \geq q = n/t$, this gives

$$\Delta(f) \geq \frac{n/t}{3n/m} = \frac{\sqrt{m}}{3}.$$

For general sizes, take $t = \lfloor \sqrt{m} \rfloor$ and balance the t groups as evenly as possible. The same probability argument gives $p \le 1/t$. Now let $q = |N_{j^\star}|$, so $n/(2t) \le q \le 2n/t$. Using the exact welfare expressions above,

$$\mathbb{E}[\mathrm{sw}_v(f(\prec))] = \frac{n-q}{m} + pq \le \frac{n}{m} + \frac{2n}{t^2}.$$

It follows that

$$\Delta(f) \ge \frac{n/(2t)}{n/m + 2n/t^2} = \frac{t}{2(t^2/m + 2)} \ge \frac{t}{6} \ge \frac{\sqrt{m}}{12}.$$

Here $t^2 \le m$ and $t \ge \sqrt{m}/2$. □

5 Randomized upper bounds stated in lecture

The lecture states that a simple randomized ordinal rule achieves

$$\Delta(f) = O\Big(\sqrt{m \log m}\Big).$$

The board also records a refined existence bound

$$\Delta(f) = O\Big(\sqrt{m \log^* m}\Big).$$

Here $\log^* m$ is the number of applications of $\log_2$ needed to reduce m to at most one. For example, $16 \mapsto 4 \mapsto 2 \mapsto 1$, so $\log^* 16 = 3$.