

CS 386 Lecture 3: Fair Division

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1 Introduction

A simple example of a Fair Division problem is cutting a cake between n players. To actually do the math we need to build a model and make some assumptions. We model the cake as a unit interval, and the following assumptions hold (here v_i is the valuation function of the i -th player):

- i) Normalization: $v_i([0, 1]) = 1 \forall i$
- ii) Positivity: $v_i(A) > 0 \forall A \neq \emptyset$
- iii) Additivity: $v_i(A \cup B) = v_i(A) + v_i(B)$ (A and B should be disjoint).
- iv) Divisibility: for any piece A and $\alpha \in (0, 1)$ there exists $B \subset A$ s.t. $v_i(B) = \alpha v_i(A)$
- v) (Optional) Continuity: v_i is continuous.

2 Types of Fairness

Now we need to define what "fairness" means. There are a few ways to do so, in particular:

- i) Proportionality: everyone gets at least $\frac{1}{n}$ in value (based on their own assessment).
- ii) Exactness: everyone gets exactly $\frac{1}{n}$ in value.
- iii) Envy-freeness: my valuation of my piece is at least my valuation of every other player's piece.
- iv) Pareto-optimality: an allocation is Pareto-optimal if I can't construct a different allocation where everyone is at least as happy as they are now with at least one person strictly happier. This is the hardest to achieve, as we will see.

3 Algorithms

3.1 The $n=2$ case

Let's start with $n = 2$. It turns out that we already know an algorithm for this case, namely "You cut, I choose" or "Divide and choose" – in this setup, player A makes a cut, and player B chooses first. We can easily see that this algorithm guarantees proportionality and envy-freeness. However, exactness is only guaranteed for player A, and Pareto-optimality fails.

There exists another algorithm for the $n = 2$ case, Austin's Moving Knives (1961). Here a knife is held above the left edge of the cake and it is moved to the right until any of the players says "Stop!" Then the player who made the stop (let's denote them as player 1) overtakes the knife and then picks up another knife and holds it above the left edge. Then they move both knives in parallel so that the valuation of the part of the cake in-between the knives for player 1 is always $1/2$. This continues until the second player says "Stop!" After that, the knives cut the cake and the second player chooses either the middle part or the outside pieces. Player 1 gets the remaining part.

This algorithm is envy-free and proportional as well, but it is also exact, which is not the case for "You cut, I choose." Another significant difference is that Austin's Moving Knives algorithm is continuous, since there exists a continuum of decisions that a player can make (e.g. scream "Stop!" at 0.2, or at 0.22, or at 0.222, and so on). This is clearly not the case in "You cut, I choose," which is a finite algorithm. Furthermore, finite algorithms can either be bounded (i.e. there exists an upper bound on the number of decisions that the players will have to make) or unbounded.

3.2 The general case

Let's now consider the $n > 2$ case. The general case is way more complicated, as we shall see. First, let's take a look at Last Diminisher (1948). Here, we first give a knife to the first player, who makes a mark at what they believe to be $1/n$ of the value from the left. Each player then has the option to either make a new mark to the left of all the previous ones or pass. Whoever made the last mark takes the slice and leaves the game, which continues until everyone is served. This clearly guarantees proportionality, since everyone gets at least $1/n$. However, nothing else is guaranteed: this cannot be envy-free since the players who leave early have no say in the distribution of the remaining part of the cake. There is a continuous version of this algorithm, which proceeds as follows: a knife is moved from the left edge to the right until someone stops it, and then they walk away with this piece of the cake.

There are several other algorithms, e.g. Lone Chooser in 1964, Lone Divider in 1943 for $n = 3$ case and in 1967 for the general case. All these algorithms guar-

antee proportionality, however they can't guarantee anything else. In particular, developing an envy-free algorithm turned out to be way harder.

3.3 History of envy-free algorithms

- 1960: Selfridge-Conway algorithm for $n = 3$ case: finite, bounded, envy-free, requires at most five cuts. Nice.
- 1980: Stromquist develops a continuous algorithm for a division with contiguous allocations for the $n = 3$ case and shows that envy-free contiguous allocations exist for any n .
- 1995-2000: Three finite algorithms (unbounded), not contiguous.
- 1999: Envy free, contiguous, might be infinite.
- 2008: Stromquist comes back and proves that no finite protocol exists for envy-free contiguous division for the general case.
- 2016: Aziz and Mackenzie: finite, bounded, envy free, worst case runtime $n^{n^{n^{n^{\dots}}}}$

4 Some Relevant Papers

1. <https://www.jstor.org/stable/2320951?seq=1> – Stromquist's 1980 paper
2. <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v15i1r11/pdf> – Stromquist's 2008 paper
3. <https://arxiv.org/abs/1604.03655> – Aziz and Mackenzie, 2016.