CS 386 – Lecture 9
Introduction to Mechanism Design: VCG Mechanisms

October 15, 2026

Reminder: the exam is next week.

1 Setup

We are given a set A of *options* (or *alternatives*). Every player $i \in \{1, \dots, n\}$ has a secret **valuation** $v_i^* \colon A \to \mathbb{R}$. Write $V = (v_1, \dots, v_n)$ for the profile of reported valuations.

Definition 1. A **mechanism** takes in all the (reported) valuation functions and decides what to do. Formally, it is a function $f \colon \mathcal{V} \to A$ together with a set of prices $p_1, \dots, p_n$ that the players pay.

The usual objective is to maximize **social welfare**, the total happiness $\sum_i v_i(a)$.

2 Some reasonable goals

1. **Welfare.** Choose $a \in A$ to maximize $\sum_i v_i(a)$.
2. **Incentive compatibility (IC).** Define the *utility* of player i as

$$U_i(a) = v_i(a) - p_i.$$

   A mechanism is **incentive compatible** if every player maximizes their utility by reporting their true valuation v_i^* (i.e., by playing honestly).
3. **Individual rationality.** Utilities should be nonnegative.
4. Try to avoid mechanisms that inherently lose money.

3 Single-item auctions

There is a single good and n bids. We must decide who gets the item and how much to charge. If we want to maximize $\sum_i v_i(a)$, we give the item to the highest bidder.

Idea 1: give it to the highest bidder and charge what they bid. Say $v_1^* = 500$ and $v_2^* = 100$. *[Scribe's note: Bidder 1 could report just above* 100*, still win, and pay less than* 500*, so honesty is not optimal and this is not IC.]*

Idea 2: give it to the highest bidder and charge the second-highest bid. This is the **Vickrey second-price auction**. Suppose i has "secret" valuation v_i^*.

- If i does not get the item by reporting v_i^*, then some player j reported $v_j > v_i^*$. If i instead reports some $v_i > v_j$, then i wins but pays $v_j > v_i^*$, so i has negative utility.
- If v_i^* is the maximum over all bids, then increasing the report to $v_i > v_j$ (where v_j is the next-highest bid) does not change anything, and going below the next-highest bid means i no longer gets the item.

4 Vickrey–Clarke–Groves (VCG) mechanisms

Definition 2. A **VCG mechanism** is any mechanism that satisfies:

1) The option it chooses always maximizes $\sum_i v_i(a)$, i.e. $f(V) \in \arg\max_{a \in A} \sum_i v_i(a)$.
2) For some functions $h_1, \ldots, h_n$, where h_i has *no dependence on* v_i, the price charged to i is

$$p_i(V) = h_i(V) - \sum_{j \neq i} v_j\big(f(V)\big), \qquad V = (v_1, \ldots, v_n).$$

The utility of player i whose true valuation is v_i^*, when the option a is chosen, is

$$U_i(a) = v_i^*(a) - p_i = v_i^*(a) + \sum_{j \neq i} v_j(a) - h_i(V)$$

and, treating i's own term as one of the summands,

$$= \sum_{j} v_j(a) - h_i(V),$$

where the last sum runs over *all* j and i's true value v_i^* is used for $j = i$. *[Scribe's note: The handwritten notes write $j \neq i$ under this last sum; it should be over all j, since the $v_i^*(a)$ term is still present.]*

Theorem 1. *Every VCG mechanism is IC.*

Proof. Let $V^* = (v_1, \ldots, v_{i-1}, v_i^*, v_{i+1}, \ldots, v_n)$ be the profile in which i is honest, and let $V = (v_1, \ldots, v_n)$ with $v_i \neq v_i^*$ be the profile in which i lies; the others report the same in both. Let $a^* = f(V^*)$ and $a = f(V)$. Then

$$U_i(a) = v_i^*(a) - p_i(V) = v_i^*(a) + \sum_{j \neq i} v_j(a) - h_i(V),$$
$$U_i(a^*) = v_i^*(a^*) - p_i(V^*) = v_i^*(a^*) + \sum_{j \neq i} v_j(a^*) - h_i(V^*).$$

Since h_i does not depend on v_i, we have $h_i(V) = h_i(V^*)$. A VCG mechanism run on V^* chooses a^* to maximize $v_i^*(a^*) + \sum_{j \neq i} v_j(a^*)$, so

$$U_i(a^*) \;\geq\; U_i(a).$$

Thus honesty is optimal for i whatever the others report. □

5 Examples

Single-item auctions with $h_i = 0$. Setting $h_i \equiv 0$ gives $p_i(V) = -\sum_{j \neq i} v_j(f(V))$. In a single-item auction, if i gets the item the others receive nothing and $p_i = 0$; otherwise $p_i = -v_j$, where j is the person who gets the item. *[Scribe's note: The notes write "v_j" for the second case; with $h_i = 0$ the sign is negative, so the losers are* paid *v_j. This mechanism is IC but loses money, which is why goal 4 matters and why we use the next rule.]*

The Clarke pivot rule. Take

$$h_i(V) = \max_{b \in A} \sum_{j \neq i} v_j(b).$$

Then

$$p_i(V) = \max_{b \in A} \sum_{j \neq i} v_j(b) - \sum_{j \neq i} v_j\big(f(V)\big) \;\geq\; 0,$$

because the maximum is at least the value at $b = f(V)$. So no player is ever paid. The utility of an honest player i is

$$\begin{aligned} U_i(V) &= v_i^*\big(f(V)\big) + \sum_{j \neq i} v_j\big(f(V)\big) - \max_{b \in A} \sum_{j \neq i} v_j(b) \\ &= \sum_{j} v_j\big(f(V)\big) - \max_{b \in A} \sum_{j \neq i} v_j(b). \end{aligned}$$

Let b attain the maximum. Assuming valuations are nonnegative, $\sum_{j \neq i} v_j(b) = \sum_j v_j(b) - v_i(b) \leq \sum_j v_j(b)$, so

$$U_i(V) \;\geq\; \sum_{j} v_j\big(f(V)\big) - \sum_{j} v_j(b) \;\geq\; 0,$$

where the last inequality holds because $f(V)$ maximizes $\sum_j v_j(\cdot)$. Hence the Clarke pivot rule is individually rational. *[Scribe's note: The final chain of inequalities in the notes is hard to read; this is my reconstruction of it, and it assumes valuations are nonnegative.]*