1. **Binary Addition is Regular!** Professor A. Tom Attah from the Massachusetts Institute of Typography is studying an interesting formal language. To begin, let our alphabet $\Sigma$ be the set of all $3 \times 1$ binary vectors:

$$\Sigma = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

A correct addition of two binary numbers can be represented by a string in $\Sigma^*$. For example,

\[
\begin{array}{c}
0 & 1 & 0 & 1 \\
+ & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 1
\end{array}
\]

would be represented by the following string of four symbols from $\Sigma$.

\[
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}
\]

The language $L$ is the set of all strings in $\Sigma^*$ representing correct additions. Show that Professor Attah’s language is regular by constructing a DFA for this language. Explain briefly what each state of your DFA represents. (Note: Remember that a DFA scans
its input tape from left to right but addition is performed from right to left. This is what makes this problem interesting! It’s possible to solve this problem with a DFA with very few states—needlessly complicated DFA’s may not receive full credit on this exercise.)

2. **DFAs and NFAs.** Let \( L \) be the language of strings \( w \) over the alphabet \( \{0, 1\} \) such that \( w \) contains the substring \( 0ab0 \) or \( 1ab1 \) where \( a, b \in \{0, 1\} \). (For example, 0000, 0010, 1111, 1011, 1101 would be such substrings.)

   (a) Draw the state diagram of a DFA that accepts \( L \).

   (b) Draw the state diagram of a NFA that accepts \( L \). Your NFA must use non-determinism and should have at most 9 states and 11 edges (an edge labeled with both 0 and 1 counts as a single edge).

3. **Yet Another Closure Property of the Regular Languages!** Show that if \( L \) is regular, then so is \( \text{HALF PALINDROME}(L) = \{x \mid xx^R \in L\} \).

   (Recall that \( w^R \) denotes the reversal of a string \( w \).) Please explain your construction carefully!

4. **Nonregular Languages.** Prove that each of the languages below is not regular, using the Distinguishability Lemma. Your proofs must be clear, complete, and rigorous. (Ideally, they will also be concise—it’s possible that a well-worded paragraph might be sufficient for a proof!)

   (a) \( L = \{0^i1^{2i} \mid i \geq 0\} \).

   (b) \( L = \{0^i1^{i+j} \mid i, j \geq 0\} \).

   (c) \( L = \{ww^R \mid w \in \{a, b, c\}^*\} \).