

CS 376 –Algorithm Design and Analysis

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Propositions

- Defn: *proposition* – a statement that has the property of truth or falsity
- Propositions are the key elements to represent, analyze, or explain declarative knowledge

Propositions:

•Washington, D.C. is the capital of the USA.

•Poughkeepsie is the capital of New York.

• $1 + 1 = 2$

• $2 + 2 = 3$

The first and third of these are true; the second and fourth are false.

Non-Propositions:

•What time is it?

•Pass the salt.

• $x + 1 = 2$

• $x*y + 5*z$

presuming values for x , y , z are not given / known

Propositional operators

- Recall: *proposition* – a statement that has the property of truth or falsity
 - Often, we use *propositional letters* (or *variables*) to represent propositions: e.g., p stands for “Poughkeepsie is the capital of NY”
- There are several *operators* (sometimes called *boolean operators*) that can construct new propositions from old ones
 - Negation* (“not”): if P is a proposition, $\neg P$ is a proposition
 - Conjunction* (“and”): P and Q
 - Disjunction* (“or”): P or Q
 - Implication* (“if – then”): if P then Q
 - Equivalence* (“is equal / equivalent to”): P iff Q
 - Equivalence can also be written as “if and only if”

Propositional operator: Negation

- Whatever the value of p , True or False, the value of proposition $\neg p$ (written $\neg p$) is the opposite
 - If p is “Today is Monday,” $\neg p$ is “It is not the case that today is Monday,” or more simply “Today is not Monday.”
- Negation can be expressed with a *truth table*

p	$\neg p$	proposition
T	F	truth values
F	T	

Propositional operator: Conjunction

- Conjunction—the “and” operator
 - Whatever the values of propositions p , q , conjunction p and q (written $p \wedge q$ or $p \&\& q$) is also a proposition
 - If p is “Today is Monday” and q is “It is snowing today,” then $p \wedge q$ is “Today is Monday and it is snowing today.”
 - $p \wedge q$ is true on snowy Mondays and false on any day that is not Monday, and on any day that is Monday but not snowing
- Conjunction values as a *truth table*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Propositional operator: Disjunction

- Disjunction—the “or” operator
 - Whatever the values of propositions p , q , disjunction p or q (written $p \vee q$ or $p \parallel q$) is also a proposition
 - If p is “Today is Monday” and q is “It is snowing today,” then $p \vee q$ is “Today is Monday or it is snowing today.”
 - $p \vee q$ is true on any day that is a Monday or on which it is snowing – including snowy Mondays (it is not *exclusive*) – and false only on days that are not Mondays on which it is not snowing
- Disjunction values as a *truth table*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The *non-exclusive* sense of “or” can be a bit subtle

Exercise: What would the *exclusive-or* operator’s truth table look like?

It turns out there *is* such an operator, and it’s commonly used in logic! The English word “or” is a complicated thing to understand!

Propositional operator: Implication

- Implication—the “if...then” operator (also called *conditional*)
 - Whatever the values of propositions p, q , implication *if p then q* (written $p \rightarrow q$) is also a proposition
 - If p is “Today is Monday” and q is “It is snowing today,” then $p \rightarrow q$ is “If today is Monday then it is snowing today.”
 - Vocabulary: in $p \rightarrow q$, p is called the *hypothesis* (or *antecedent*) and q is called the *conclusion* (or *consequent*)
- Implication values as a *truth table*

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Really? *These* are the truth values for implication?

They look like the values for $(\neg p \vee q)$! (**Exercise:** Check for yourselves!!)

Sounds if-y: *Material Implication*

- Meaning for implication symbol $\rightarrow$ in propositional logic is referred to as *material implication*
 - It says that $p \rightarrow q$ is False exactly when p is True and q is False
 - Not the same as every meaning of “if...then” in English, but it’s what’s used in logic

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Examples of material implication and natural language usage:

•Politician says: “If I am elected, then I will fix the environment”

•False if the speaker is elected and doesn’t fix the environment

•True if, e.g., the speaker doesn’t get elected

•“If today is Friday, then $2 + 2 = 4$ ”

•True *no matter what day it is*

•“If today is Friday, then $2 + 2 = 5$ ”

•True except on Fridays, even though $2 + 2 = 5$ is false!

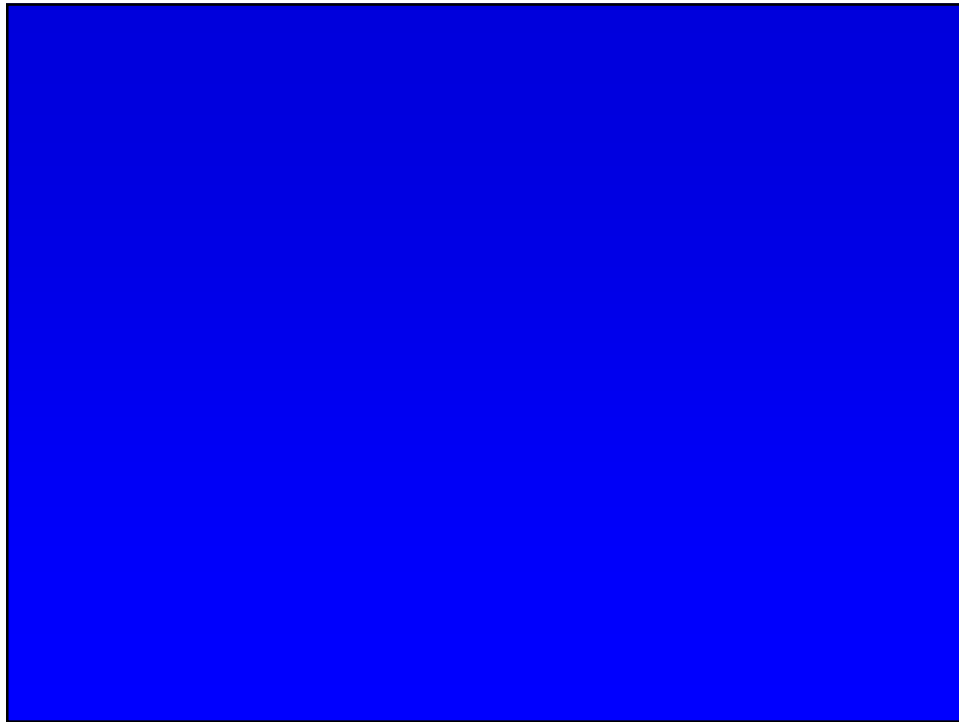
Properties of operators

- Logical operators have an *order of operations* just like mathematical operators
 - From high to low: negation; conjunction; disjunction; implication
 - Conjunction is kinda like multiplication; disjunction is kinda like addition
 - Math: $-k * (x + y)$
 - Logic: $\neg p \wedge (q \vee r)$
- Also similarly, disjunction and conjunction are *commutative* and *associative*
 - Associative: e.g., $p \wedge q \wedge r$ is $(p \wedge q) \wedge r$ is $p \wedge (q \wedge r)$
 - Commutative: e.g., $p \wedge q$ is $q \wedge p$
 - similar with disjunction
- Implication is *right-associative*
 - $p \rightarrow q \rightarrow r$ is $p \rightarrow (q \rightarrow r)$

The *biconditional* (or *equivalence*) operator

- The *biconditional* (or *equivalence*) operator:
 - If p and q are propositions, then $p \leftrightarrow q$ is a proposition, read as “ p if and only if q ”
 - $p \leftrightarrow q$ is true exactly when p and q have the same truth values
- What does the truth table for $\leftrightarrow$ look like?
- How could we define the biconditional in terms of operators we already know (not, and, or, if... then)?

The equivalence operator can also be written as $\equiv$ or $==$ in other contexts.



Exercise: Evaluating *boolean* expressions

- Defn: Propositions are *boolean*-valued expressions—i.e., their values are either True or False
- Boolean expressions are evaluated like any other mathematical expressions

Examples: Let $p = \text{True}$, $q = \text{False}$, $r = \text{True}$. What do the following expressions evaluate to?

1. $p \wedge \neg r$
2. $q \vee \text{False}$
3. $p \rightarrow q$
4. $q \leftrightarrow p$
5. $q \leftrightarrow \neg \text{True}$
6. $r \vee (p \wedge q)$
7. $(p \vee r) \rightarrow ((p \vee q) \wedge r)$
8. $\text{True} \rightarrow r$

Compound propositions and their truth tables

- Just as we use truth tables to understand meanings of propositional operators, we can also use them to understand compound propositions
 - The truth table for $(p \vee \neg q) \rightarrow (p \wedge q)$:

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Exercise: Truth tables for compound propositions

- What are truth tables for the following propositions?
 1. $p \rightarrow \neg p$
 2. $p \leftrightarrow \neg p$
 3. $(p \rightarrow q) \wedge (\neg p \rightarrow q)$
 4. $(p \vee q) \wedge r$
 5. $p \rightarrow (\neg q \vee r)$