

CS375 Weekly Assignment 1

Due Feb 17th 10PM ET

A general note for CS375: When writing up your homework, please write neatly and explain your answers clearly, giving all details needed to make your answers easy to understand. Graders may not award full credit to incomplete or illegible solutions.

You are encouraged to discuss the problems with your classmates, **but your submission should be your own**. When writing your solutions up, you should not be looking at any other student's submission.

Clear communication **is the point**, on every assignment. In general in CS375, unless explicitly specified otherwise, answers should be accompanied by explanations. Answers without explanations may not receive full credit. Please feel free to ask me any questions about explanations that might come up!

Necessary acknowledgements: many of these questions are reproductions or adaptations of questions of Eric Aaron's devising. Others are from our course texts as listed on the syllabus.

1. Pick 2 of the following and prove them using induction.

i) $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

ii) $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$

iii) $\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$ ($\leftarrow$ we're going to use this one in class later)

iv) $\sum_{i=1}^n (2i-1) = n^2$

v) $\sum_{i=1}^n 2i = n(n+1)$

vi) $\sum_{i=1}^n F_i = F_{n+2} - 1$ where F_i is the i^{th} Fibonacci number ($F_1 = F_2 = 1$, $F_{i>2} = F_{i-1} + F_{i-2}$)

vii) $n! > 2^n$ for $n \geq 4$ ($\leftarrow$ useful for problem 3)

Some Solutions

(a) $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Base Case: $\sum_{i=1}^1 i^2 = 1 = \frac{1(1+1)(2+1)}{6}$

Inductive Hypothesis: Suppose $\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$ for some value k

Inductive Step:

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \sum_{i=1}^k i^2 \\ &= (k+1)^2 + \frac{k(k+1)(2k+1)}{6} \\ &= \frac{k+1}{6} (6(k+1) + k(2k+1)) \\ &= \frac{k+1}{6} (2k^2 + 7k + 6) \\ &= \frac{k+1}{6} (k+2)(2k+3) \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}\end{aligned}$$

Which shows the formula holds for the next value $(k+1)$, completing the induction.

(b) $n! > 2^n$ for $n \geq 4$

Base Case ($n = 4$): $4! = 24 > 16 = 2^4$

Inductive Hypothesis: Suppose $k! > 2^k$ for some value $k \geq 4$

Inductive Step:

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)2^k \\ &\geq 52^k \\ &> 22^k \\ &= 2^{k+1}\end{aligned}$$

Which shows the inequality holds for the next value $(k+1)$, completing the induction.

2. Use induction to argue that the following algorithm correctly sorts a list:

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SelectionSort(A[0:n]):
    for i in range(n):
        min_index = i
        for j in range(i+1, n):
            if A[j] < A[min_index]:
                min_index = j
        if min_index != i:
            swap A[i] with A[min_index]
```

Solution: I'm going to show that after iteration i , we will have that $A[0 : (i+1)]$ (meaning we include index i but not $i+1$) will contain the $i+1$ smallest items of A in sorted order. Note that because we never change the contents of A as an array, this means that after iteration $i = n-1$ the array $A[0 : n]$ will be sorted. In particular, I'll use induction to show it.

Base Case ($i = 0$): During iteration $i = 0$, the algorithm finds the item of minimal value and swaps it to index 0. Hence, at the end of this first iteration, the $A[0]$ is now the smallest item in A and $A[0 : 1]$ is now sorted (note that this range is only one index).

Inductive Hypothesis: Suppose we know that after some iteration i our claim holds.

Inductive Step: Let's show that it also holds after iteration $i+1$: during iteration $i+1$, the algorithm finds the smallest item in the subarray from $A[i+1 :]$ and moves it to index $i+1$. By induction, all smaller items have already been found and are in $A[0 : (i+1)]$ in sorted order, so now $A[0 : (i+2)]$ is now also sorted and contains the $i+2$ smallest items.

3. Pick 15 of the following functions and list them in order of asymptotic growth rate from least to greatest. Eg if the only functions were n^3 , 1 , n^2 , n , then the correct order would be 1 , n , n^2 , n^3 , as $1 = O(n)$, $n = O(n^2)$, and $n^2 = O(n^3)$. For any two that you pick that are Θ of each other, point them out. Note that $\lg$ is shorthand for $\log_2$. Of the 15 that you pick, you must pick: n^2 , $n!$, n^3 , $\lg^2 n$, $\lg n$, $2^{\lg n}$, $4^{\lg n}$, $(n+1)!$, $\sqrt{n}$, n , 2^n , $n \lg n$ (which means you should pick 3 more on top of these).

$\lg(\lg^* n)$	$2^{\lg^* n}$	$(\sqrt{2})^{\lg n}$	n^2	$n!$	$(\lg n)!$
$(\frac{3}{2})^n$	n^3	$\lg^2 n$	$\lg(n!)$	2^{2^n}	$n^{\frac{1}{\lg n}}$
$\lg \lg n$	$\lg^* n$	$n2^n$	$n^{\lg \lg n}$	$\lg n$	1
$2^{\lg n}$	$(\lg n)^{\lg n}$	e^n	$4^{\lg n}$	$(n+1)!$	$\sqrt{n}$
$\lg^*(\lg n)$	$2^{\sqrt{2}^{\lg n}}$	n	2^n	$n \lg n$	$2^{2^{n+1}}$

We almost certainly won't use $\lg^*$ this semester, and I can't imagine many (if any) of you have seen it before, but $\lg^*(n)$ = the number of times needed to compose $\lg$ to get to a value less than 1. Eg: $\lg^* 2 = 1$, $\lg^* 4 = 2$, $\lg^* 16 = 3$, $\lg^* 65536 = 4$, $\lg^*(2^{65536}) = 5$. It is an EXTREMELY slowly growing function.

You should show your work for each sequential function you write.

Solution: I'm going to show it for the 12 required - let me know if you have any questions about the other ones. The correct order is:

$$1, \lg n, \lg^2 n, \sqrt{n}, n, 2^{\lg n}, n \lg n, n^2, 4^{\lg n}, n^3, 2^n, n!, (n+1)!$$

i) 1 vs $\lg n$:

$$\lim_{n \rightarrow \infty} \frac{1}{\lg n} = 0.$$

ii) $\lg n$ vs $\lg^2 n$:

$$\lim_{n \rightarrow \infty} \frac{\lg n}{\lg^2 n} = \lim_{n \rightarrow \infty} \frac{1}{\lg n} = 0.$$

iii) $\lg^2 n$ vs $\sqrt{n}$: As a reminder, $\log_b(n) = \frac{\log_c(n)}{\log_c(b)}$. The main takeaway from this is that **all logs of a constant base are within a constant factor of each other**. So switching between $\log$, $\lg$, and $\ln$ will make no difference (if they are the outside function) from an asymptotic analysis perspective. So for the sake of making the math easier I'm going to treat $\lg^2 n$ and $\ln^2 n$. For that,

$$\lim_{n \rightarrow \infty} \frac{\ln^2 n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{2 \ln n \cdot \frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{4 \ln n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{4 \cdot \frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n \rightarrow \infty} \frac{8}{\sqrt{n}} = 0.$$

iv) $\sqrt{n}$ vs n : note that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0.$$

v) n vs $2^{\lg n}$: note that 2^n and $\lg n$ are *inverse functions* of each other. Thought of in a different way, $\lg n$ computes the power by which you would raise 2 to get to n ; we are raising 2 to that power. Hence $2^{\lg n} = n$, so $n = \Theta(2^{\lg n})$ (I should hope it's clear that if $f(n) = g(n)$ then $f(n) = \Theta(g(n))$).

vi) n vs $n \lg n$:

$$\lim_{n \rightarrow \infty} \frac{n}{n \lg n} = \lim_{n \rightarrow \infty} \frac{1}{\lg n} = 0$$

vii) $n \lg n$ vs n^2 :

$$\lim_{n \rightarrow \infty} \frac{n \lg n}{n^2} = \lim_{n \rightarrow \infty} \frac{\lg n}{n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

viii) n^2 vs $4^{\lg n}$: Note that $4^{\lg n} = (2^2)^{\lg n} = 2^{2 \lg n} = (2^{\lg n})^2 = n^2$, so $n^2 = \Theta(4^{\lg n})$.

ix) n^2 vs n^3 :

$$\lim_{n \rightarrow \infty} \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

x) n^3 vs 2^n :

$$\lim_{n \rightarrow \infty} \frac{n^3}{2^n} = \lim_{n \rightarrow \infty} \frac{3n^2}{2^n \ln 2} = \lim_{n \rightarrow \infty} \frac{6n}{2^n \ln^2 2} = \lim_{n \rightarrow \infty} \frac{6}{2^n \ln^3 2} = 0.$$

xi) 2^n vs $n!$: So for this, we could use 1.vii to identify that

$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} \leq \lim_{n \rightarrow \infty} \frac{n!}{n!} = \lim_{n \rightarrow \infty} 1 = 1 < \infty$$

and be done with it, but technically this doesn't rule out them being Θ of each other. If we wanted to do that, we could observe that

$$\lim_{n \rightarrow \infty} \frac{2^n}{n!} \leq \lim_{n \rightarrow \infty} \frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{2} \cdot \frac{2}{1} \leq \lim_{n \rightarrow \infty} \frac{2}{n} \cdot \frac{2}{n-1} \cdots \frac{2}{\frac{n}{2}+1} \cdot \frac{2}{\frac{n}{2}} \leq \lim_{n \rightarrow \infty} \left(\frac{4}{n}\right)^{\frac{n}{2}},$$

where the inside of those parentheses is heading to zero, so the whole limit is rapidly heading to 0.

xii) $n!$ vs $(n+1)!$:

$$\lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$