

CS375 Weekly Assignment 5

Due Mar 17th 10PM ET

A general note for CS375: When writing up your homework, please write neatly and explain your answers clearly, giving all details needed to make your answers easy to understand. Graders may not award full credit to incomplete or illegible solutions.

You are encouraged to discuss the problems with your classmates, **but your submission should be your own**. When writing your solutions up, you should not be looking at any other student's submission.

Clear communication **is the point**, on every assignment. In general in CS375, unless explicitly specified otherwise, answers should be accompanied by explanations. Answers without explanations may not receive full credit. Please feel free to ask me any questions about explanations that might come up!

Necessary acknowledgements: many of these questions are reproductions or adaptations of questions of Eric Aaron's devising. Others are from our course texts as listed on the syllabus.

1. In 1854, archaeologists discovered Sumerians clay tablets, carved around 2000 BCE, that list the squares of integers up to 59. This discovery led some scholars to conjecture that ancient Sumerians performed multiplication by reduction to squaring, using an identity like $x \times y = x^2 + y^2 - (x - y)^2$. Unfortunately, those same scholars are silent on how the Sumerians supposedly squared larger numbers. Four thousand years later, we can finally rescue these Sumerian mathematicians from their lives of drudgery through the power of recursion!
 - (a) Describe a variant of Karatsuba's algorithm that squares any n -digit number in $\Theta(n^{\lg 3})$ time (by reducing it to squaring three $\frac{n}{2}$ digit numbers). (Karatsuba actually did this in 1960.) Use the master theorem to verify that your runtime is correct.
 - (b) Describe a recursive algorithm that squares any n digit number in $\Theta(n^{\lg_3(6)})$ time. As a bit of a hint, if you are trying to compute x^2 where $x = 10^{\frac{2n}{3}}a + 10^{\frac{n}{3}}b + c$ and a , b , and c are all $\frac{n}{3}$ digits in length, try computing:
 - i. a^2
 - ii. b^2
 - iii. c^2
 - iv. $(a + b)^2$
 - v. $(b + c)^2$
 - vi. $(a + c)^2$and then putting them together in a nice way. Use the master theorem or a recursion tree to verify the correctness of your runtime.
 - (c) Unfortunately, $\lg 3 < \lg_3(6)$, so we've made negative progress. Let's fix that by squaring in $\Theta(n^{\lg_3(5)})$ time. Using the same notation as above, compute
 - i. a^2
 - ii. $(4a + 2b + c)^2$
 - iii. c^2
 - iv. $(a + b + c)^2$
 - v. $(a - b + c)^2$

and put the terms together nicely. Use the master theorem or a recursion tree to verify the correctness of your runtime.¹

2. Suppose we are given a set $S = \{x_1 = (v_1, w_1), \dots, x_n = (v_n, w_n)\}$ of n items where v_i denotes the value of x_i and w_i denotes the weight of x_i . For any element x , let $S_{<x}$ be the set of items of less value than x and let $S_{>x}$ be the set of items of more value than x . For any subset R of S , let $w(R)$ denote the sum of the weights of elements in R . We'll say a **weighted median** of S is *any* element x in S that satisfies $w(S_{<x}) \leq \frac{w(S)}{2}$ and $w(S_{>x}) \leq \frac{w(S)}{2}$.

Describe and analyze a recursive algorithm for computing a weighted median of a set in $O(n)$ time.

3. For the following functions of T , pick 2 more (that you didn't do last week) and determine their asymptotic complexity **by means of a recursion tree**. After, confirm your answer to these 2 **and the 3 from last week** by means of applying the Master Theorem, **or state that the Master Theorem doesn't apply to the chosen function**. You can assume that $T(n) = 1$ for all $n < 10$ (or any reasonable constant) and that all relevant mathematical operations drop remainders (so $\frac{3}{2} = \sqrt{2} = 1$, etc). Note that I have purposely shuffled the list so that the order of these is not in increasing difficulty.

(a) $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\lg n}$

(l) $T(n) = T(n-1) + 1$

(b) $T(n) = 8T\left(\frac{n}{2}\right) + n^3\sqrt{n}$

(m) $T(n) = 7T\left(\frac{n}{2}\right) + n^2 \lg n$

(c) $T(n) = 4T\left(\frac{n}{2}\right) + n^2 \lg n$

(n) $T(n) = T(n-2) + n^2$

(d) $T(n) = 5T\left(\frac{n}{3}\right) + \frac{n}{\lg n}$

(o) $T(n) = 2T\left(\frac{n}{4}\right) + \sqrt{n}$

(e) $T(n) = \sqrt{n}T(\sqrt{n}) + n$

(p) $T(n) = 16T\left(\frac{n}{4}\right) + n^2$

(f) $T(n) = 2T\left(\frac{n}{2}\right) + n \lg n$

(q) $T(n) = 2T\left(\frac{n}{2}\right) + n^3$

(g) $T(n) = T\left(\frac{n}{2}\right) + T\left(\frac{n}{4}\right) + T\left(\frac{n}{8}\right) + n$

(r) $T(n) = 8T\left(\frac{n}{3}\right) + n^2$

(h) $T(n) = \sqrt{2n}T(\sqrt{2n}) + \sqrt{n}$

(s) $T(n) = T(n-1) + n$

(i) $T(n) = T(n-2) + \frac{1}{\lg n}$

(t) $T(n) = T(n-1) + \lg n$

(j) $T(n) = T\left(\frac{8n}{11}\right) + n$

(u) $T(n) = 2T(n-1) + 1$

(k) $T(n) = 3T\left(\frac{n}{3}\right) + \frac{n}{\lg n}$

(v) $T(n) = T(n-1) + \frac{1}{n}$

I feel like this assignment is already a bit long so I am somewhat disinclined to put anymore problems on here. However, Erickson's chapter on recursion is a literal treasure trove of great practice problems. CLRS maybe a little less so for our class, as there's a lot of material that it would rely on that I haven't taught. If you happen to have *Algorithm Design* by Kleinberg and Tardos, that text also has some nice problems. As always, feel free to come to me with any problems you think about and would like advice on.

¹This was originally going to be your exam question :)